INTRODUCTORY TREATISE ON LIE'S THEORY OF FINITE CONTINUOUS TRANSFORMATION GROUPS

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PREFACE

IN this treatise an attempt is made to give, in as elementary a form as possible, the main outlines of Lie's theory of Continuous Groups. I desire to acknowledge my great indebtedness to Engel's three standard volumes on this subject; they have been constantly before me, and but for their aid the present work could hardly have been undertaken. His *Continuierliche Gruppen*, written as it was under Lie's own supervision, must always be referred to for the authoritative exposition of the theory in the form in which Lie left it. During the preparation of this volume I have consulted the several accounts which Scheffers has given of Lie's work in the books entitled *Differential-gleichungen*, *Continuierliche Gruppen*, and the *Berührungs-Transformationen*; and also the interesting sketch of the subject given by Klein in his lectures on Higher Geometry. In addition to these I have read a number of original memoirs, and would specially refer to the writings of Schur in the *Mathematische Annalen* and in the *Leipziger Berichte*. Yet, great as are my obligations to others, I am not without hope that even those familiar with the theory of Continuous Groups may find something new in the form in which the theory is here presented. Within the limits of a volume of moderate size the reader will not expect to find an account of all parts of the subject. Thus the theory of the possible types of group-structure has been omitted. This branch of

group-theory has been considerably advanced by the labours of others than Lie; especially by W. Killing, whose work is explained and extended by Cartan in his *Thèse sur la structure des groupes de transformations finis et continus*[1]. A justification of the omission of this part of the subject from an elementary treatise may perhaps also be found in the fact that it does not seem to have yet arrived at the completeness which characterizes other parts of the theory.

The following statement as to the plan of the book may be convenient. The first chapter is introductory, and aims at giving a general idea of the theory of groups. The second chapter contains elementary illustrations of the principle of extended point transformation. Chapters III–V establish the fundamental theorems of group-theory. Chapters VI and VII deal with the application of the theory to complete systems of linear partial differential equations of the first order. Chapter VIII discusses the invariant theories associated with groups. Chapter IX considers the division of groups into certain great classes. Chapter X considers when two groups are transformable, the one into the other. Chapter XI deals with isomorphism. Chapters XII and XIII show how groups are to be constructed when the structure constants are given. Chapter XIV discusses Pfaff's equation and the integrals of non-linear partial differential equations of the first order. Chapter XV considers the theory of complete systems of homogeneous functions. Chapters XVI–XIX explain the theory of contact transformations. Chapter XX deals

[1] See the article on Groups by Burnside in the *Encyclopaedia Britannica*.

with the theory of Differential Invariants. Chapters XXI–XXIV show how all possible types of groups can be obtained when the number of variables does not exceed three. Chapter XXV considers the relation subsisting between the systems of higher complex numbers and certain linear groups. I have added a fairly full table of contents, a reference to which will, I think, make the general drift of the theory more easily grasped by the reader to whom the subject is new.

It now remains to express my gratitude to two friends for the great services which they have rendered me during their reading of the proof-sheets. Mr. H. T. Gerrans, Fellow of Worcester College, Oxford, at whose suggestion this work was undertaken, found time in the midst of many pressing engagements to aid me with very helpful criticism. Mr. H. Hilton, Fellow of Magdalen College, Oxford, and Mathematical Lecturer in the University College of North Wales, has most generously devoted a great deal of time to repeated corrections of the proofs, and suggested many improvements of which I have gladly availed myself. With the help thus afforded me by these friends I have been able to remove some obscurities of expression and to present the argument in a clearer light, though I fear I must still ask the indulgence of my readers in many places. Finally I desire to thank the Delegates of the Oxford University Press for undertaking the publication of the book, and the staff of the Press for the great care which they have taken in printing it.

J. E. CAMPBELL.

HERTFORD COLLEGE, OXFORD.
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ERRATUM

Page 62, line 14, for Y_s read Y_r

CHAPTER I

DEFINITIONS AND SIMPLE EXAMPLES OF GROUPS

§ 1. If we have two sets of variables, $x_1, \ldots, x_n$ and $x_1', \ldots, x_n'$, connected by the equations

$$(1) \qquad x_i' = f_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

they will define a transformation scheme, provided that we can solve the equations so as to express the variables $x_1, \ldots, x_n$ in terms of the variables $x_1', \ldots, x_n'$.

We shall denote the transformation scheme (1) by S.

The operation, which consists in substituting for $x_1, \ldots, x_n$ in any function of these variables $f_1, \ldots, f_n$ respectively, will be denoted by S_x, or simply by S when there is no need to indicate the objects on which the operation S is performed.

So S_y will denote the operation of substituting for $y_1, \ldots, y_n$ respectively, $f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n)$ respectively.

Similarly the operation which consists in substituting for x_i the function $f_i(f_1, \ldots, f_n)$ will be denoted by S^2, and so on.

Solving the equations (1) we obtain the algebraically equivalent set

$$(2) \qquad x_i = F_i(x_1', \ldots, x_n'), \qquad (i = 1, \ldots, n).$$

From (1) and (2) we see that

$$x_i \equiv F_i(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$$
$$\equiv f_i(F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)).$$

We therefore denote the scheme (2) by S^{-1}, and the operation of substituting $F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n$ respectively by S_x^{-1}.

The two schemes (1) and (2) are said to be inverse to one another.

§ 2. If we have a second transformation scheme T, viz.

$$x_i' = \phi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

then TS_x will denote the operation of substituting $f_i(\phi_1, \ldots, \phi_n)$ for x_i.

The function $f_i(\phi_1, \ldots, \phi_n)$ may be more compactly written $f_i\phi$, the function $f_i(\phi_1(\psi_1, \ldots, \psi_n), \ldots, \phi_n(\psi_1, \ldots, \psi_n))$ may be written $f_i\phi\psi$, and so on.

In TS the order in which the operations are to be taken is from right to left; but it should be noticed that, f being the functional symbol which corresponds to S, and ϕ the functional symbol which corresponds to T, the functional symbol which corresponds to TS is not ϕf but $f\phi$.

So if we have a third transformation scheme U, viz.

$$x_i' = \psi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

UTS would denote the operation which consists in first operating with S, then operating with T on this result, and finally operating with U; the functional symbol which córresponds to UTS is $f\phi\psi$: that is, UTS is the operation which consists in substituting $f_1\phi\psi, \ldots, f_n\phi\psi$ for $x_1, \ldots, x_n$ respectively*.

ST denotes the operation of substituting $\phi_1 f, \ldots, \phi_n f$ for $x_1, \ldots, x_n$ respectively, and TS the operation of substituting for $x_1, \ldots, x_n$ respectively, $f_1\phi, \ldots, f_n\phi$; if then

$$f_i\phi = \phi_i f, \qquad (i = 1, \ldots, n),$$

$ST = TS$, and the operations S and T are said to be *permutable.*

§ 3. In accordance with what precedes, STS_x^{-1} denotes the operation of replacing x_i by $F_i\phi f$; it follows therefore that when STS_x^{-1} is applied to $f_i(x_1, \ldots, x_n)$ this function becomes $f_iF\phi f$; that is, since $f_iF \equiv x_i$, it becomes $\phi_i(f_1, \ldots, f_n)$.

We thus see that the operation STS^{-1} has the same effect on the variables $x_1', \ldots, x_n'$, when expressed in terms of $x_1, \ldots, x_n$ by the scheme S, viz.

$$x_i' = f_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

as the operation $T_{x'}$ has on the variables $x_1', \ldots, x_n'$; STS^{-1} is therefore said to be an operation *similar* to T with respect to S.

§ 4. If we have a system of transformation schemes $S_1, S_2, \ldots$, and if the resultant operation generated by successively performing any two operations of the system is itself an operation of the system, then the transformation schemes are said to form a *group.*

* In Burnside's *Theory of Groups* the order of operations is taken from left to right. The reason why we have adopted the opposite convention is that we shall deal chiefly with differential operators, and it would violate common usage to write $\frac{d}{dx}y$ in the form $y\frac{d}{dx}$.

§ 5. A group is said to be *continuous* when, if we take any two operations of the group S and T, we can always find a series of operations within the group, of which the effect of the first of the series differs infinitesimally from the effect of S; the effect of the second differs infinitesimally from the effect of the first; the third from the second and so on; and, finally, the effect of the last of the series differs infinitesimally from T. Naturally this series must contain an infinite number of operations unless S and T should themselves chance to differ only infinitesimally.

§ 6. If the equations which define the transformation schemes $S_1, S_2, \ldots$ of a group involve arbitrary functional symbols the group is said to be an *infinite* group; but we shall see that a group, with an infinite number of operations within it, is not necessarily an infinite group.

§ 7. A group is said to be *discontinuous* if it contains no two operations whose effects differ only infinitesimally.

It should be noticed that the two classes of continuous and discontinuous groups, though mutually exclusive, do not exhaust all possible classes of transformation groups.

An example of a transformation group which belongs to neither of the above classes is

$$x' = \omega x + a,$$

where a is a parameter and ω any root of $x^m = 1$.

A series of transformations within the group, the effects of consecutive members of which only differ infinitesimally, could be placed between

$$x' = \omega x + a \quad \text{and} \quad x' = \omega x + b,$$

viz. $$x' = \omega x + a + \frac{b-a}{n}, \quad x' = \omega x + a + \frac{2(b-a)}{n}, \ldots,$$
$$x' = \omega x + a + \frac{n-1}{n}(b-a),$$

where n is a very large integer; but such a series could not be placed between

$$x' = \omega x + a \quad \text{and} \quad x' = \omega' x + b$$

if ω and ω' are different m^{th} roots of unity.

§ 8. The transformation scheme

$$x_i' = x_i, \qquad (i = 1, \ldots, n)$$

is called the *identical transformation*; if it is included in the transformations of a group, the group is said to *contain* the identical transformation.

§ 9. A simple example of a discontinuous group is the set of six transformations,

$$x' = x,\ x' = \frac{1}{1-x},\ x' = \frac{x-1}{x},\ x' = \frac{1}{x},\ x' = 1-x,\ x' = \frac{x}{x-1},$$

by which the six anharmonic ratios of four collinear points are interchanged amongst themselves.

If we denote the six corresponding operations by S_1 (which is equal to unity since it transforms x into x), S_2, S_3, S_4, S_5, S_6 respectively, we verify the statement that these operations form a group when we prove that $S_2 S_3 = S_1$, $S_4 S_5 = S_3$, and so on.

Inversion with respect to a fixed circle offers an even simpler example of a discontinuous group; it only contains two operations, viz. the identical operation S_1 and the operation S_2 which consists in replacing x by $\dfrac{a^2 x}{x^2+y^2}$ and y by $\dfrac{a^2 y}{x^2+y^2}$ when the circle of inversion is $x^2+y^2 = a^2$.

The group property follows from the fact that $S_2^2 = S_1$.

§ 10. In the above two examples there are only a finite number of operations in the group; the set of transformations,

$$x' = \alpha x + \beta y, \quad y' = \gamma x + \delta y,$$

where α, β, γ, δ are any positive integers, is an example of a discontinuous group with an infinite number of operations.

The group property follows from the fact that from

$$x' = \alpha x + \beta y, \quad y' = \gamma x + \delta y,$$

and

$$x'' = px' + qy', \quad y'' = rx' + sy',$$

where p, q, r, s are another set of integers, we can deduce

$$x'' = (p\alpha + q\gamma)x + (p\beta + q\delta)y, \quad y'' = (r\alpha + s\gamma)x + (r\beta + s\delta)y,$$

where the coefficients of x and y are still positive integers.

§ 11. Simple examples of continuous groups are the following:

(1) $$x' = f(x), \quad y' = \phi(y)$$

where f and ϕ are arbitrary functional symbols; the group property follows from the fact that these equations and

$$x'' = \lambda(x'), \quad y'' = \mu(y')$$

where λ and μ are other arbitrary functional symbols, lead to

$$x'' = \lambda f(x), \quad y'' = \mu\phi(y).$$

(2) $$x' = f(x, y), \quad y' = \phi(x, y), \quad z' = \psi(z)$$

where f, ϕ, and ψ are all arbitrary functional symbols.

(3) $$x' = f(x, y), \quad y' = \phi(x, y)$$

where f and ϕ are conjugate functions; for if θ and ψ are two other conjugate functions, and

$$x'' = \theta(x', y'), \quad y'' = \psi(x', y'),$$

then $$x'' + iy'' = F(x' + iy') = F\Phi(x + iy),$$

so that x'' and y'' are also conjugate functions of x and y; that is, the transformation system, which is obviously continuous, has the group property.

(4) $$x' = f(x, y, z), \quad y' = \phi(x, y, z), \quad z' = \psi(x, y, z)$$

where f, ϕ, ψ are functions of their arguments such that their Jacobian

$$\frac{\partial(f, \phi, \psi)}{\partial(x, y, z)} \equiv 1.$$

The group property follows from the identity

$$\frac{\partial(x'', y'', z'')}{\partial(x, y, z)} \equiv \frac{\partial(x'', y'', z'')}{\partial(x', y', z')} \cdot \frac{\partial(x', y', z')}{\partial(x, y, z)}.$$

These are examples of *infinite continuous groups*, for the transformation schemes in (1), (2), (3), (4) involve arbitrary functional symbols.

§ 12. If the transformation scheme

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

defines a group; that is, if from the equations

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r),$$
$$x_i'' = f_i(x_1', \ldots, x_n', b_1, \ldots, b_r)$$

we can deduce $x_i'' = f_i(x_1, \ldots, x_n, c_1, \ldots, c_r)$,

where $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are two sets of r unconnected arbitrary constants, and $c_1, \ldots, c_r$ are constants connected with these two sets, then this group is said to be *finite and continuous*.

If values of $a_1, \ldots, a_r$ can be found such that

$$x_i \equiv f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

the group contains the identical transformation; if $a_1^0, \ldots, a_r^0$ are these values, $a_1^0, \ldots, a_r^0$ are said to be the parameters of the identical transformation. Finite continuous groups do exist which do not contain the identical transformation, but the properties of such groups will not be investigated here.

§ 13. A transformation whose effect differs infinitesimally from the identical transformation is said to be an *infinitesimal transformation.* The general form of such a transformation is

$$x_i' = x_i + t\xi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$$

where t is a constant so small that its square may be neglected.

If $\phi(x_1, \ldots, x_n)$ is any function of $x_1, \ldots, x_n$, then if we expand $\phi(x_1', \ldots, x_n')$ in powers of t, neglecting terms of the order t^2, we get

$$\begin{aligned}\phi(x_1', \ldots, x_n') &= \phi(x_1 + t\xi_1, \ldots, x_n + t\xi_n) \\ &= \phi(x_1, \ldots, x_n) + t\left(\xi_1 \frac{\partial\phi}{\partial x_1} + \ldots + \xi_n \frac{\partial\phi}{\partial x_n}\right).\end{aligned}$$

If then we let X denote the linear operator,

$$\xi_1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \ldots + \xi_n(x_1, \ldots, x_n)\frac{\partial}{\partial x_n},$$

$$\phi(x_1', \ldots, x_n') = (1 + tX)\,\phi(x_1, \ldots, x_n),$$

so that we take $\qquad 1 + tX$

to be the symbol of an infinitesimal transformation; and we call X the infinitesimal operator, or simply the operator, which corresponds to this infinitesimal transformation.

We shall see that any transformation whatever of a finite continuous group which contains the identical transformation can be obtained by indefinite repetition of an infinitesimal operation; that is we shall prove that if

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

are the equations of such a group,

$$f_i(x_1, \ldots, x_n, a_1, \ldots, a_r) \equiv \text{the limit of } \left(1 + \frac{X}{m}\right)^m x_i,$$

when m is made infinite, and X is some linear operator.

This limit is, we know by ordinary algebra,

$$\left(1 + \frac{1}{1!}X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \ldots\right)x_i.$$

§ 14. A simple example of a finite continuous group is the projective transformation of the straight line

$$x' = \frac{a_1 x + a_2}{a_3 x + a_4}$$

where a_1, a_2, a_3, a_4 are four arbitrary constants; the group

property of these transformation schemes can be easily verified.

In this group four arbitrary constants appear, but only three *effective parameters*, viz. the ratios of these constants; it is always to be understood that the parameters of a group are taken to be effective; thus, if a_1 and a_2 always occurred in the combination a_1+a_2 they would be replaced by the single effective parameter a_1.

The identical transformation in the above projective group is found by taking the parameters $a_2=a_3=0$ and $a_1=a_4$.

If we take $a_1=a_4(1+e_2)$, $a_2=e_1a_4$, $a_3=-e_3a_4$, where e_1, e_2, e_3 are small constants whose squares may be neglected,

$$x'=\frac{(1+e_2)x+e_1}{1-e_3x}=x+e_1+e_2x+e_3x^2.$$

This is the general form of an infinitesimal transformation of the projective group of the straight line.

§ 15. If $\quad x_i'=x_i+e_k\xi_{ki}(x_1,\ldots,x_n), \quad \binom{i=1,\ldots,n}{k=1,\ldots,r}$

are a set of r infinitesimal transformations, they are said to be *independent* if no set of r constants, $\lambda_1,\ldots,\lambda_r$, not all zero, can be found such that

$$\lambda_1\xi_{1i}+\ldots+\lambda_r\xi_{ri}\equiv 0, \qquad (i=1,\ldots,n).$$

The r linear operators, $X_1,\ldots,X_r$, where

$$X_k\equiv\xi_{k1}\frac{\partial}{\partial x_1}+\ldots+\xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k=1,\ldots,r),$$

are said to be *independent* when no r constants, $\lambda_1,\ldots,\lambda_r$, not all zero, can be found such that

$$\lambda_1X_1+\ldots+\lambda_rX_r\equiv 0.$$

Any linear operator which can be expressed in the form

$$\lambda_1X_1+\ldots+\lambda_rX_r$$

is said to be *dependent* on $X_1,\ldots,X_r$.

If we have r operators, $X_1,\ldots,X_r$, such that no identical relation of the form

$$\psi_1X_1+\ldots+\psi_rX_r\equiv 0$$

connects them, where $\psi_1,\ldots,\psi_r$ are r functions of the variables $x_1,\ldots,x_n$, not all zero, they are said to be *unconnected* operators. It is necessary to distinguish between independent operators and unconnected operators; unconnected operators are neces-

sarily independent, but independent operators are not necessarily unconnected ; thus

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}$$

are unconnected operators, but X, Y, Z where

$$X \equiv y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y \equiv z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z \equiv x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

are three connected operators, since

$$xX + yY + zZ \equiv 0,$$

and yet they are independent.

In the projective group of the straight line there are three independent operators, viz.

$$\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x},$$

but only one unconnected operator.

We shall find that there are always just as many independent operators in a group as there are effective parameters.

§ 16. If X_1 and X_2 are any two linear operators, the symbol $X_1 X_2$ means that we are first to operate with X_2 and then with X_1; the symbol $X_1 X_2$ is not then itself a linear operator; but $X_1 X_2 - X_2 X_1$ is such an operator, since the parts in $X_1 X_2$ and $X_2 X_1$ which involve such terms as $\frac{\partial^2}{\partial x_1 \partial x_2}$, are the same in both.

The expression $X_1 X_2 - X_2 X_1$ is written (X_1, X_2) and is called the *alternant* of X_1 and X_2.

In the projective group of the straight line we see that

$$\left(\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}\right) \equiv \frac{\partial}{\partial x},$$

$$\left(\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x}\right) \equiv 2x\frac{\partial}{\partial x},$$

$$\left(x\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x}\right) \equiv x^2\frac{\partial}{\partial x},$$

so that the alternant of any two of the three infinitesimal operators of the group is dependent on these three operators. This will be proved to be a general property of the infinitesimal operators of any finite continuous group.

§ 17. The most general infinitesimal operator of the projective group of the straight line is X where

$$X \equiv (e_1 + e_2 x + e_3 x^2) \frac{d}{dx}$$

and e_1, e_2, e_3 are arbitrary constants.

If we take

(1) $y = 2(4e_1e_3 - e_2^2)^{-\frac{1}{2}} \tan^{-1} \{(4e_1e_3 - e_2^2)^{-\frac{1}{2}} (2e_3x + e_2)\}$,

it is easily verified that

$$\frac{d}{dy} = (e_1 + e_2x + e_3x^2)\frac{d}{dx} = X;$$

and therefore

$$\left(1 + \frac{1}{1!}X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \dots\right)x$$

is equal to

$$\left(1 + \frac{1}{1!}\frac{d}{dy} + \frac{1}{2!}\frac{d^2}{dy^2} + \dots\right)\left(\sqrt{\frac{e_1}{e_3} - \frac{e_2^2}{4e_3^2}} \tan \frac{\sqrt{4e_1e_3 - e_2^2}}{2} \cdot y - \frac{e_2}{2e_3}\right);$$

and this by Taylor's theorem is equal to

$$\sqrt{\frac{e_1}{e_3} - \frac{e_2^2}{4e_3^2}} \tan \frac{\sqrt{4e_1e_3 - e_2^2}}{2}(y+1) - \frac{e_2}{2e_3}.$$

If we substitute for y its value in terms of x we shall have an expression of the form

$$\frac{a_1x + a_2}{a_3x + a_4}$$

where a_1, a_2, a_3, a_4 are functions of e_1, e_2, e_3; and we thus verify, for the case of the projective group of the straight line, the general theorem that any transformation of a group can be obtained by repeating indefinitely a properly chosen infinitesimal transformation.

§ 18. If we have two groups

$$x_i' = f_i(x_1, \dots, x_n,\ a_1, \dots, a_r), \qquad (i = 1, \dots, n)$$

and $$y_i' = \phi_i(y_1, \dots, y_m,\ a_1, \dots, a_r), \qquad (i = 1, \dots, m)$$

where m and n are integers not necessarily equal; and if we have a correspondence between $S_{a_1, \dots, a_r}$ the operations of the first, and $T_{a_1, \dots, a_r}$ the operations of the second such that to every operation $S_{a_1, \dots, a_r}$ a single operation $T_{a_1, \dots, a_r}$ corresponds, and to every operation $T_{a_1, \dots, a_r}$ a single operation

$Sa_1, \dots, a_r$ and to the product $Sa_1, \dots, a_r \; Sb_1, \dots, b_r$ the product $Ta_1, \dots, a_r \; Tb_1, \dots, b_r$, then the two groups are said to be *simply isomorphic*.

It might appear at first that any two groups with the same parameters would be simply isomorphic; we could of course say that $Sa_1, \dots, a_r$ corresponds uniquely to $Ta_1, \dots, a_r$ and $Sb_1, \dots, b_r$ to $Tb_1, \dots, b_r$, but it would not follow that $Sa_1, \dots, a_r \; Sb_1, \dots, b_r$ corresponded to $Ta_1, \dots, a_r \; Tb_1, \dots, b_r$. For from the definition of the group

$$Sa_1, \dots, a_r \; Sb_1, \dots, b_r = Sc_1, \dots, c_r,$$

where $c_1, \dots, c_r$ are functions of the two sets $a_1, \dots, a_r$ and $b_1, \dots, b_r$; and these functions will naturally depend upon the forms of the functions $f_1, \dots, f_n$ which defined the first group; while from

$$Ta_1, \dots, a_r \; Tb_1, \dots, b_r = T\gamma_1, \dots, \gamma_r,$$

where $\gamma_1, \dots, \gamma_r$ are functions of $a_1, \dots, a_r$ and $b_1, \dots, b_r$, whose forms depend on the forms of the functions $\phi_1, \dots, \phi_m$, we could not in general conclude that $\gamma_1 = c_1, \dots, \gamma_r = c_r$ unless the two groups are specially related.

An example of two simply isomorphic groups is offered by

$$x_1' = a_1 x_1 + a_1 a_2 x_2, \quad x_2' = a_1 x_2$$

and

$$y_1' = y_1 + a_2 y_2 + \log a_1, \quad y_2' = y_2.$$

If we take two operations of the first

$$x_1' = a_1 x_1 + a_1 a_2 x_2, \quad x_2' = a_1 x_2,$$

$$x_1'' = b_1 x_1' + b_1 b_2 x_2', \quad x_2'' = b_1 x_2',$$

we deduce

$$x_1'' = c_1 x_1 + c_1 c_2 x_2, \quad x_2'' = c_1 x_2,$$

where

$$c_1 = a_1 b_1, \quad c_2 = a_2 + b_2,$$

so that the group property of the first is verified.

Taking two operations of the second

$$y_1' = y_1 + a_2 y_2 + \log a_1, \quad y_2' = y_2,$$

$$y_1'' = y_1' + b_2 y_2' + \log b_1, \quad y_2'' = y_2',$$

we also deduce

$$y_1'' = y_1 + c_2 y_2 + \log c_1, \quad y_2'' = y,$$

where

$$c_1 = a_1 b_1, \quad c_2 = a_2 + b_2,$$

and thus verify the group property of the second and its simple isomorphism with the first.

§ 19. Returning now to the definition of a finite continuous group and writing $f_i(x_1, \dots, x_n, a_1, \dots, a_r)$ in the abridged form $f_i(x, a)$ we see that if

$$x_i' = f_i(x, a), \quad x_i'' = f_i(x', b),$$

then
$$x_i'' = f_i(x, c),$$

where $\quad c_k = \phi_k(a_1, \dots, a_r, b_1, \dots, b_r), \qquad (k = 1, \dots, r).$

It will now be proved that these functions $\phi_1, \dots, \phi_r$ define two groups, one of which is simply isomorphic with the given group.

It is to be assumed that f_i is an analytic function of $x_1, \dots, x_n, a_1, \dots, a_r$ within the region of the arguments $x_1, \dots, x_n$ $a_1, \dots, a_r$; and also that the parameters are effective; that is if we suppose f_i expanded in powers of $x_1, \dots, x_n$ the coefficients will be analytic functions of $a_1, \dots, a_r$, and there will be exactly r such functionally unconnected coefficients in terms of which all other coefficients can be expressed.

From the group definition we have

$$f_i(x, c) = x_i'' = f_i(x', b) = f_i\left(f_1(x, a), \dots, f_n(x, a), b_1, \dots, b_r\right),$$

and since the parameters are effective we have

$$(1) \quad c_k = \phi_k(a_1, \dots, a_r, b_1, \dots, b_r), \qquad (k = 1, \dots, r).$$

Also
$$x_i = F_i(x', a), \qquad (i = 1, \dots, n)$$

being the inverse transformation scheme to

$$x_i' = f_i(x, a),$$

we have

$$f_i(x', b) = f_i(x, c) = f_i\left(F_1(x', a), \dots, F_n(x', a), c_1, \dots, c_r\right);$$

and therefore if we expand $f_i(x', b)$ in powers and products of $x_1', \dots, x_n'$, since there are exactly r parameters involved, we see that in the expansion of

$$(2) \qquad f_i\left(F_1(x', a) \dots F_n(x', a), c_1, \dots, c_r\right)$$

there must be exactly r unconnected coefficients.

We further see that b_k can in general be expressed in terms of $a_1, \dots, a_r$, $c_1, \dots, c_r$ subject to certain limitations in the values which $a_1, \dots, a_r$, $c_1, \dots, c_r$ can assume in order that (2) may remain an analytic function of its arguments.

Thus suppose we have the equations

$$f(x, y) = \alpha, \quad \phi(x, y) = \beta,$$

a necessary condition that we may be able to express x and y in terms of α, β is that the Jacobian of the functions $f(x, y)$

and $\phi(x, y)$ should not vanish identically, or as we shall say the functions must be unconnected. The form of the functions f and ϕ may, however, be such that whatever the values of x and y, real or complex, f cannot exceed an assigned value a, nor ϕ an assigned value b; the equations

$$f(x, y) = \alpha, \quad \phi(x, y) = \beta$$

could not then be solved unless $\alpha \leqq a$ and $\beta \leqq b$.

When we come to seek the conditions that a group may contain the identical transformation we shall have to make $a_k = c_k$, and the result may be that we cannot solve the equations (1), and in this case the group will not contain the identical transformation.

In general, however, we can express b_k in terms of $a_1, \ldots, a_r$, $c_1, \ldots, c_r$, and therefore in the equations

$$c_k = \phi_k(a_1, \ldots, a_r, b_1, \ldots, b_r), \qquad (k = 1, \ldots, r)$$

the functional forms $\phi_1, \ldots, \phi_r$ are such that the determinant

$$\begin{vmatrix} \dfrac{\partial \phi_1}{\partial b_1} & \cdot & \cdot & \cdot & \dfrac{\partial \phi_1}{\partial b_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial \phi_r}{\partial b_1} & \cdot & \cdot & \cdot & \dfrac{\partial \phi_r}{\partial b_r} \end{vmatrix}$$

cannot vanish identically.

Similarly from $x_i'' = f_i(x', b)$ we deduce $x_i' = F_i(x'', b)$; and from $x_i' = f_i(x, a)$ and from these identities we have

$$f_i(x, a) = F_i(x'', b) = F_i\big(f_1(x, c), \ldots, f_n(x, c), b_1, \ldots, b_r\big);$$

so that we see that a_k can be expressed in terms of $b_1, \ldots, b_r$, $c_1, \ldots, c_r$ and conclude that the determinant

$$\begin{vmatrix} \dfrac{\partial \phi_1}{\partial a_1} & \cdot & \cdot & \cdot & \dfrac{\partial \phi_1}{\partial a_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial \phi_r}{\partial a_1} & \cdot & \cdot & \cdot & \dfrac{\partial \phi_r}{\partial a_r} \end{vmatrix}$$

cannot vanish identically.

We can therefore conclude that the equations

$$(3) \qquad y_k' = \phi_k(y_1, \ldots, y_r, a_1, \ldots, a_r), \qquad (k = 1, \ldots, r)$$

define a transformation scheme with r effective parameters,

and we shall now prove that these are the equations of a group.

We have $f_i(x', b) = x_i'' = f_i(x, c) = f_i(x, \phi(a, b))$;

and if we take any other set of parameters $\gamma_1, \ldots, \gamma_r$,

$$x_i''' = f_i(x'', \gamma) = f_i(x', \phi(b, \gamma)) = f_i(x, \phi(a, \phi(b, \gamma))).$$

Now $\quad f_i(x'', \gamma) = f_i(x, \phi(c, \gamma)) = f_i(x, \phi(\phi(a, b), \gamma)),$

so that by equating the coefficients in these two expressions for $f_i(x'' \gamma)$ we have the identity

$$\phi_k(a, \phi(b, \gamma)) \equiv \phi_k(\phi(a, b), \gamma).$$

This identity leads at once to the group property of (3), for by its aid we deduce from

$$y_k' = \phi_k(y, a) \text{ and } y_k'' = \phi_k(y', b) = \phi_k(\phi(y, a), b)$$

that $$y_k'' = \phi_k(y, \phi(a, b)),$$

that is the equations (3) generate a group which is known as the *first parameter group* of

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n).$$

It is an obvious property of this parameter group to be its own parameter group.

From the definition of simple isomorphism we see that two groups are then, and only then, simply isomorphic when they have the same parameter group; the first parameter group is therefore simply isomorphic with the group of which it is the first parameter group.

§ 20. In exactly the same way we see that the equations

$$y_k' = \phi_k(a_1, \ldots, a_r, y_1, \ldots, y_r), \qquad (k = 1, \ldots, r)$$

are the equations of a group.

This group is called the *second parameter group*; it is its own second parameter group; but it is not isomorphic with the original group; for from $y_k' = \phi_k(a, y)$, $y_k'' = \phi_k(b, y')$ we deduce $y_k'' = \phi_k(c, y)$, where $c_k = \phi_k(b_1, \ldots, b_r, a_1, \ldots, a_r)$, and $\phi_k(b, a)$ is not generally equal to $\phi_k(a, b)$.

The two parameter groups are such that any operation of the first is permutable with any operation of the second.

This comes at once from the fundamental identity

$$\phi_k(a, \phi(b, c)) \equiv \phi_k(\phi(a, b), c),$$

which is true for all values of the suffix k and the arbitrary parameters $a_1, \ldots, a_r$, $b_1, \ldots, b_r$, $c_1, \ldots, c_r$; for to prove that

$$y_k' = \phi_k(y, a) \text{ and } y_k' = \phi_k(b, y)$$

are permutable operations it is only necessary to prove that

$$\phi_k(\phi(b, y), a) \equiv \phi_k(b, \phi(y, a)).$$

§ 21. As an example we shall find the first parameter group of the general linear homogeneous group,

$$x_i' = \Sigma a_{hi} x_h,$$

the summation being for all positive integral values of h from 1 to n inclusive.

As such summations will very frequently occur it is necessary to employ certain conventions to express them. The subscripts will always denote positive integers; those which vary in the summation will be supposed to go through all positive integral values between their respective limits, thus in

$$\Sigma c_{\alpha\beta j} \lambda_{\beta i} \lambda_{\alpha k},$$

where the summation is for all positive integral values of α from p to r inclusive, and for all positive integral values of β from q to k inclusive, we should indicate the sum by

$$\sum_{\alpha=p,\, \beta=q}^{\alpha=r,\, \beta=k} c_{\alpha\beta j} \lambda_{\beta i} \lambda_{\alpha k}.$$

When the two limits are the same we should write the above sum in the form

$$\sum_{\alpha=\beta=p}^{\alpha=\beta=k} c_{\alpha\beta j} \lambda_{\beta i} \lambda_{\alpha k}.$$

This would not of course mean that $\alpha = \beta$ throughout the summation; a summation in which $\alpha = \beta$ would be expressed by

$$\sum_{\alpha=p}^{\alpha=k} c_{\alpha\alpha j} \lambda_{\alpha i} \lambda_{\alpha k}.$$

When the lower limit is unity it will be omitted, thus when $p = 1$ the sum would be written

$$\sum^{\alpha=\beta=k} c_{\alpha\beta j} \lambda_{\beta i} \lambda_{\alpha k}.$$

Expressing the linear group in this notation from

$$x_i' = \sum^{h=n} a_{hi} x_h \quad \text{and} \quad x_i'' = \sum^{h=n} b_{hi} x_h',$$

we obtain $$x_i'' = \sum^{h=n} c_{hi} x_h,$$

where $$c_{hi} = \sum^{k=n} a_{hk} b_{ki}.$$

If then $y_{hi}, \ldots$ are n^2 variables, the linear group

$$y'_{hi} = \sum^{k=n} a_{ki} y_{hk}$$

is the first parameter group of the general linear homogeneous group in n variables.

It will be noticed that this group is itself a linear homogeneous group in n^2 variables, but it is of course not the general linear group in n^2 variables.

The second parameter group is

$$y'_{hi} = \sum^{k=n} a_{hk} y_{ki}.$$

§ 22. If in any given group

$$(1) \quad x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

we pass to a new set of variables $y_1, \ldots, y_n$ where

$$(2) \qquad y_i = g_i(x_1, \ldots, x_n),$$

and to a cogredient set $y_1', \ldots, y_n'$ given by

$$y_i' = g_i(x_1', \ldots, x_n'),$$

where $g_1, \ldots, g_n$ are any n unconnected functions of their arguments, we must obtain equations of the form

$$(3) \quad y_i' = \phi_i(y_1, \ldots, y_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n).$$

We are now going to find the relation between the two transformation schemes (1) and (3).

Let T denote the operation which replaces x_1 by g_1, x_2 by g_2, and so on.

If then $$x_i = G_i(y_1, \ldots, y_n)$$

is the inverse scheme to (2), T^{-1} will denote the operation which replaces x_i by G_i.

We now take S_a to be the operation which replaces x_i by $f_i(x, a)$ and S_b the operation which replaces x_i by $f_i(x, b)$.

The operation TS_aT^{-1} acting on y_i that is on $g_i(x_1, \ldots, x_n)$ will transform it into y_i'; for

$$TS_aT^{-1}g_i(x_1, \ldots, x_n) = TS_a g_i(G_1, \ldots, G_n) = TS_a x_i,$$

and $TS_a x_i = Tf_i(x_1, \ldots, x_n,\ a_1, \ldots, a_r) = f_i(g_1, \ldots, g_n,\ a_1, \ldots, a_r)$,

and $f_i(g_1, \ldots, g_n,\ a_1, \ldots, a_r) = f_i(y_1, \ldots, y_n,\ a_1, \ldots, a_r) = y_i'$.

The operations of the transformation schemes (3) are therefore $\quad TS_aT^{-1},\quad TS_bT^{-1}, \ldots$

and since $\quad TS_aT^{-1}\,TS_bT^{-1} = TS_aS_bT^{-1} = TS_cT^{-1}$,

we see that the equations (3) are the equations of a group simply isomorphic with the group (1). The two groups (1) and (3) are said to be *similar*. Similar groups are therefore simply isomorphic, but it is not true conversely that all simply isomorphic groups are similar. The necessary and sufficient conditions for the similarity of groups are obtained in Chapter X. It will then be seen why it is not possible to transform the two isomorphic groups given in § 18 into one another. Groups which are similar are also said to be of the same *type*.

§ 23. It will be proved later that groups which contain the identical transformation can have their operations arranged in pairs which are inverse to one another; that is to every transformation S_a another transformation S_b of the group will correspond in such a way that the product of the two will be the identical transformation. If then T is any operation within the group, T^{-1} will also be an operation of the group, and so will the operation TST^{-1}. This operation is said to be *conjugate* to S with respect to T; if TST^{-1} is equal to S, whatever operation of the group T may be, then S is permutable with every operation of the group and is said to be an *Abelian operation.*

If T is an operation of the group so is TST^{-1}; but even if T is not such an operation, TST^{-1} may be an operation of the given group: we should then say that T was an operation which transformed the group into itself.

If T_1 and T_2 are two operations each of which transforms a given group into itself, then $T_1ST_1^{-1}$ is an operation within the group; $T_2T_1ST_1^{-1}T_2^{-1}$ must then be within the group; that is, since $T_1^{-1}T_2^{-1} = (T_2T_1)^{-1}$, T_2T_1 is also an operation which transforms the given group into itself.

It follows therefore that the totality of operations with the property of transforming the group into itself, or as we shall say the totality of operations which the group *admits*, form a group. This group, however, need not be finite.

§ 24. If out of all the operations of a group a set be taken not including all the operations of the group, this set may itself satisfy the group condition; in this case it is said to be a *sub-group* of the given group.

Let $S_1, S_2, \ldots, T_1, T_2, \ldots$ be the operations of a group, and suppose that $S_1, S_2, \ldots$ form a sub-group, then $T_k S_1 T_k^{-1}$, $T_k S_2 T_k^{-1}, \ldots$ which (§ 22) is a similar group to $S_1, S_2, \ldots$ is said to be *conjugate* to the sub-group $S_1, S_2, \ldots$. Sub-groups which are conjugate to one another are also said to be of the *same type.*

If, whatever the operation T_k may be within the group $S_1, S_2, \ldots, T_1, T_2, \ldots$ the sub-group $T_k S_1 T_k^{-1}, T_k S_2 T_k^{-1}, \ldots$ coincides with $S_1, S_2, \ldots$, then the sub-group $S_1, S_2, \ldots$ is said to be a *self-conjugate sub-group.* It will be noticed that it is not necessary in order that the sub-group may be self-conjugate, that $T_k S_h T_k^{-1}$ should be identical with S_h, but only that it should be some operation of the system $S_1, S_2, \ldots$.

A group such that all its operations are commutative is called an *Abelian group.*

It is easily proved that if a group contains Abelian operations they form an *Abelian sub-group.*

Example. The linear homogeneous transformation schemes

$$x_i' = \sum^{h=n} a_{hi} x_h, \qquad (i = 1, \ldots, n),$$

where the parameters are subject to the single condition

$$\begin{vmatrix} a_{11} & \cdot & \cdot & \cdot & a_{n1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1n} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix} = 1,$$

form a group with $(n^2 - 1)$ effective parameters.

If S_a is a transformation included in this scheme, and M_a the above determinant, then, S_b being any other transformation of the scheme and M_b the determinant which corresponds to it, the determinant of $S_a S_b$ is $M_a M_b$; and therefore, since this is unity, the transformations generate a group. This group is called the *special linear homogeneous group*; it is a sub-group of the general linear homogeneous group. It is also self-conjugate within it; for if T is any operation of the general group, the determinant of $TS_a T^{-1}$ is the same as that of S_a, and therefore $TS_a T^{-1}$ is itself an operation of the special linear group.

Example. The projective group of the straight line

$$x' = \frac{a_1 x + a_2}{a_3 x + a_4}$$

contains the sub-group

$$x' = a_1 x + a_2.$$

This sub-group contains two sub-groups, viz.

$$x' = ax \quad \text{and} \quad x' = x + a;$$

the first is the homogeneous linear group, and the second is the translation group.

We shall prove later that these are the only types of finite continuous groups of the straight line; that is, all other groups of the straight line are transformable to one of these by the method of § 22; it will also be proved that every group which contains only one parameter is of the type

$$x' = x + a,$$

that is, the type of the translation group of the straight line.

§ 25. A group which contains r effective parameters is said to be of *order* r, or to be an r-fold group. We now write down some groups of transformations of the plane.

The eight-fold *projective group* is

$$x' = \frac{a_{11} x + a_{21} y + a_{31}}{a_{13} x + a_{23} y + a_{33}}, \quad y' = \frac{a_{12} x + a_{22} y + a_{32}}{a_{13} x + a_{23} y + a_{33}}.$$

The identical transformation is obtained by taking

$$a_{11} = a_{22} = a_{33},$$

and making the other parameters zero; the eight infinitesimal operators (§ 13) are then found to be

$$\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ x\frac{\partial}{\partial x}, \ y\frac{\partial}{\partial y}, \ x\frac{\partial}{\partial y}, \ y\frac{\partial}{\partial x},$$

$$x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \ xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.$$

The projective group has as a sub-group the *general linear group*, viz.

$$x' = a_{11} x + a_{21} y + a_{31}, \quad y' = a_{12} x + a_{22} y + a_{32},$$

of which the infinitesimal operators are

$$\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ x\frac{\partial}{\partial x}, \ y\frac{\partial}{\partial y}, \ x\frac{\partial}{\partial y}, \ y\frac{\partial}{\partial x}.$$

One sub-group of the general linear group is the group of movements of a rigid lamina in a plane, viz.

$$x' = x\cos\theta + y\sin\theta + a_1, \quad y' = -x\sin\theta + y\cos\theta + a_2,$$

a_1, a_2, and θ being the arbitrary parameters.

The identical transformation is obtained by putting

$$a_1 = a_2 = \theta = 0,$$

and the infinitesimal transformations by taking a_1, a_2, θ to be small unconnected constants; the infinitesimal operators are

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Each of these sub-groups could be obtained from the projective group by connecting the parameters of the latter by certain equations; thus the general linear group was obtained by taking $a_{13} = a_{23} = 0$. It must not, however, be supposed that if we are given a group, and connect its parameters by some arbitrarily chosen equation, the resulting transformation system will generally be a sub-group; this would only be true for equations of a particular form connecting the parameters of the given group.

It has been stated that there are no groups of the straight line which are not types of the projective group of the line, or of one of its sub-groups. In space of more than one dimension, however, groups do exist which are not of the projective type; thus in the plane the equations

$$x' = \frac{a_1 x + a_2}{a_3 x + a_4}, \quad y' = \frac{yx^r + a_5 x^r + a_6 x^{r-1} + \ldots + a_{r+5}}{(a_1 x + a_2)^r},$$

where the constants are arbitrary, define a non-projective group of order $r+4$. The group property may be verified easily. The identical transformation is obtained by taking $a_2 = a_3 = a_5 = \ldots = 0$, and $a_1 = a_4 = 1$, and the infinitesimal operators may be written down without much difficulty; but, since a general method of obtaining these will soon be investigated, we shall not now consider these operators.

This group is not similar to the projective group, nor to any of its sub-groups.

§ 26. In three-dimensional space many of the groups have long been known; there is the *general projective group* of order 15, viz.

$$x' = \frac{a_{11}x + a_{21}y + a_{31}z + a_{41}}{a_{14}x + a_{24}y + a_{34}z + a_{44}}, \quad y' = \frac{a_{12}x + a_{22}y + a_{32}z + a_{42}}{a_{14}x + a_{24}y + a_{34}z + a_{44}},$$

$$z' = \frac{a_{13}x + a_{23}y + a_{33}z + a_{43}}{a_{14}x + a_{24}y + a_{34}z + a_{44}}.$$

From this we obtain the *linear group* of order 12 by taking $a_{14} = a_{24} = a_{34} = 0$; the *linear homogeneous group* of order 9 by further taking $a_{41} = a_{42} = a_{43} = 0$; the *special linear homogeneous group* of order 8 by taking

$$\begin{vmatrix} a_{11}, & a_{21}, & a_{31} \\ a_{12}, & a_{22}, & a_{32} \\ a_{13}, & a_{23}, & a_{33} \end{vmatrix} = 1.$$

Other sub-groups of the general projective group are: the group of rotations about a fixed point of order 3; the group of translations, also of order 3; and the six-fold group of movements of a rigid body, obtained by combining these two groups of order 3.

There are very many other sub-groups of the projective group, but we have now perhaps given a sufficient number of examples of projective groups in three-dimensional space.

From these groups others could be deduced by transformations of the variables, but they would not be new types, thus the groups

$$x' = a_{11}x + a_{21}y + a_{31}z, \quad y' = a_{12}x + a_{22}y + a_{32}z,$$
$$z' = a_{13}x + a_{23}y + a_{33}z,$$

and
$$x' = \frac{a_{11}x + a_{21}y + a_{31}}{a_{13}x + a_{23}y + a_{33}}, \quad y' = \frac{a_{12}x + a_{22}y + a_{32}}{a_{13}x + a_{23}y + a_{33}},$$
$$z' = a_{13}xz + a_{23}yz + a_{33}z^2$$

are of the same type, for the first can be transformed into the second by the scheme

$$x_1 = xz, \quad y_1 = yz, \quad z_1 = z.$$

§ 27. We may apply the theory of groups to obtain, in terms of Euler's three angles, the formulae for the transformation from one set of orthogonal axes to another.

Describe a sphere of unit radius with the origin O as centre, and let the first set of axes intersect this sphere in A, B, C.

By a rotation ψ about the axis OC we obtain the quadrantal triangle CPQ, and a point whose coordinates referred to the first set of axes were x, y, z will, when referred to the new set, have the coordinates x', y', z' where

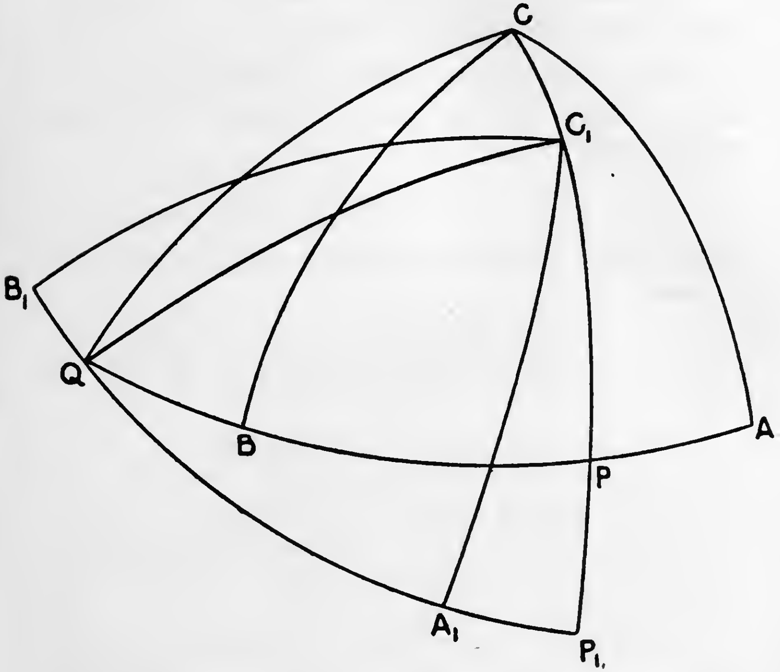
$$x' = x\cos\psi + y\sin\psi, \quad y' = -x\sin\psi + y\cos\psi, \quad z' = z.$$

By a rotation θ about OQ we pass to the quadrantal triangle C_1P_1Q, and a point with the coordinates x, y, z will now have the coordinates x'', y'', z'', where

$$x'' = x'\cos\theta - z'\sin\theta, \quad y'' = y', \quad z'' = x'\sin\theta + z'\cos\theta.$$

Finally by a rotation ϕ about OC_1 we pass to the axes OC_1, OA_1, OB_1 referred to which the coordinates of x, y, z will be x''', y''', z''', where

$$x''' = x''\cos\phi + y''\sin\phi,\quad y''' = -x''\sin\phi + y''\cos\phi,\quad z''' = z''.$$



If then R denotes the operation of replacing x, y, z respectively by

$$x\cos\psi + y\sin\psi,\quad -x\sin\psi + y\cos\psi,\quad z,$$

S the operation of replacing x, y, z by

$$x\cos\theta - z\sin\theta,\quad y,\quad x\sin\theta + z\cos\theta,$$

and T the operation of replacing x, y, z by

$$x\cos\phi + y\sin\phi,\quad -x\sin\phi + y\cos\phi,\quad z,$$

the coordinates of a point x, y, z, with respect to the first axes, will be obtained when referred to the new axes OA_1, OB_1, OC_1, by operating on x, y, z with RST, and therefore

$$x''' = (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\,x$$
$$+ (\cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi)\,y - \sin\theta\cos\phi\,.\,z,$$
$$y''' = -(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\,x$$
$$+ (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\,y + \sin\theta\sin\phi\,.\,z,$$
$$z''' = \sin\theta\cos\psi\,.\,x + \sin\theta\sin\psi\,.\,y + \cos\theta\,.\,z.$$

These are Euler's formulae; if we take

$$\phi + \psi = \epsilon_1, \quad \theta\cos(\phi - \psi) = \epsilon_2, \quad \theta\sin(\phi - \psi) = \epsilon_3,$$

and then make ϵ_1, ϵ_2, ϵ_3 small, we obtain the three infinitesimal operators

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

of this group. These can, however, be more easily obtained otherwise.

§ 28. An example of a group in three-dimensional space, which is not derivable from the projective groups by a transformation of coordinates, is

$$x' = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad y' = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3},$$
$$z' = \frac{(b_2c_3 - b_3c_2)x + (a_2b_3 - a_3b_2)(y + xz) + a_2c_3 - a_3c_2}{(b_1c_3 - b_3c_1)x + (a_1b_3 - a_3b_1)(y - xz) + a_1c_3 - a_3c_1}.$$

If we notice that

$$y' - x'z' = \frac{(b_1c_2 - b_2c_1)\,x + (a_1b_2 - a_2b_1)(y - xz) + a_1c_2 - a_2c_1}{(b_1c_3 - b_3c_1)\,x + (a_1b_3 - a_3b_1)(y - xz) + a_1c_3 - a_3c_1},$$

it will not be difficult to verify the group-property.

As the number of variables increases the number of different types of groups increases rapidly. Thus there are only three types of groups of the straight line; there are a considerable number of types of groups in the plane, but they are now all known and will be given later on; in three-dimensional space there are a very large number of types, most of which have been enumerated in Lie's works; but in space of higher dimensions no attempt has been made to exhaust the types.

CHAPTER II

ELEMENTARY ILLUSTRATIONS OF THE PRINCIPLE OF EXTENDED POINT TRANSFORMATIONS

§ 29. Some classes of differential equations have the property of being unaltered when we transform to certain new variables. Such transformation schemes obviously generate a group; for if S and T are two operations which transform the equation into itself, or as we shall say operations *admitted* by the given equation, TS will also be an operation admitted by the equation, and therefore S and T must be operations of a group. This group, however, is not necessarily finite or continuous.

The differential equation of all straight lines in the plane, viz. $\frac{d^2y}{dx^2} = 0$, is an equation of this class; for from its geometrical meaning we know that it must be unaltered by any projective transformation.

Again the differential equation of circles in a plane, viz.

$$3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{d^3y}{dx^3},$$

must admit the group of movements of a lamina in a plane, and also inversion.

It would be easy to write down many equations which, from their geometrical interpretation, must obviously admit known groups; but more equations exist admitting groups than we could always obtain by this *a priori* method; and we shall now therefore briefly consider a method by which the form of those differential expressions may be obtained which are unaltered, save for a factor, by the transformations of a known group. The method will be more fully explained and illustrated in the chapter on Differential Invariants.

§ 30. In this investigation the underlying principle is that of the *extended point transformation.*

To explain this principle let

$$x' = x + t\xi(x, y), \quad y' = y + t\eta(x, y)$$

be an infinitesimal transformation; then

$$\frac{dy'}{dx'} = \frac{\dfrac{dy}{dx} + t\left(\dfrac{\partial\eta}{\partial x} + \dfrac{\partial\eta}{\partial y}\dfrac{dy}{dx}\right)}{1 + t\left(\dfrac{\partial\xi}{\partial x} + \dfrac{\partial\xi}{\partial y}\dfrac{\partial y}{\partial x}\right)}$$

$$= \frac{dy}{dx} + t\left(\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial y}\frac{dy}{dx} - \frac{\partial\xi}{\partial x}\frac{dy}{dx} - \frac{\partial\xi}{\partial y}\left(\frac{dy}{dx}\right)^2\right),$$

since t is a constant so small that its square may be neglected.

If we denote $\dfrac{dy}{dx}$ by p, and $\dfrac{dy'}{dx'}$ by p', and the expression

$$\frac{\partial\eta}{\partial x} + \left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right)p - \frac{\partial\xi}{\partial y}p^2$$

by π, we have proved that

$$p' = p + t\pi.$$

Similarly we have

$$\frac{dp'}{dx'} = \frac{\dfrac{dp}{dx} + t\left(\dfrac{\partial\pi}{\partial x} + \dfrac{\partial\pi}{\partial y}p + \dfrac{\partial\pi}{\partial p}\dfrac{dp}{dx}\right)}{1 + t\left(\dfrac{\partial\xi}{\partial x} + \dfrac{\partial\xi}{\partial y}p\right)}$$

$$= \frac{dp}{dx} + t\left(\frac{\partial\pi}{\partial x} + p\frac{\partial\pi}{\partial y} - \frac{\partial\xi}{\partial x}\frac{dp}{dx} - \frac{\partial\xi}{\partial y}p\frac{dp}{dx} + \frac{\partial\pi}{\partial p}\frac{dp}{dx}\right).$$

If we now write r for $\dfrac{dp}{dx}$ this gives, after some easy reduction,

$$r' = r + t\rho,$$

where

$$\rho = \frac{\partial^2\eta}{\partial x^2} + \left(2\frac{\partial^2\eta}{\partial x\,\partial y} - \frac{\partial^2\xi}{\partial x^2}\right)p + \left(\frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\,\partial y}\right)p^2 - \frac{\partial^2\xi}{\partial y^2}p^3$$
$$- 3\frac{\partial\xi}{\partial y}pr + \left(\frac{\partial\eta}{\partial y} - 2\frac{\partial\xi}{\partial x}\right)r.$$

The infinitesimal transformation is said to be once *extended* when to the transformation scheme

$$x' = x + t\xi, \qquad y' = y + t\eta$$

we add $$p' = p + t\pi;$$

it is said to be twice *extended* when we add to these

$$r' = r + t\rho,$$

and so on.

A general rule for extending a point transformation to any order will be explained in Chapter XX.

We have only considered the extension of an infinitesimal transformation, but any transformation could be similarly extended; the infinitesimal transformations with their extensions are, however, the most important in seeking differential equations which admit the operations of a known group.

It will be proved in Chapter XX that if we have a group of transformations, and extend it any number of times, the resulting set of transformations will belong to a group which is simply isomorphic with the given group.

§ 31. In order to illustrate the theory of extended point transformations we shall find the absolute differential invariant of the second order; that is, an expression of the form $f(x, y, p, r)$, which is unaltered by the transformations of the group of movements of a rigid lamina in the plane xy.

In this problem the infinitesimal transformation is

$$x' = x + t\xi, \quad y' = y + t\eta, \quad p' = p + t\pi, \quad r' = r + t\rho,$$

where

$$\xi = a + cy, \quad \eta = b - cx, \quad \pi = -c(1 + p^2), \quad \rho = -3cpr,$$

and a, b, c are constants.

Since $$f(x, y, p, r) \equiv f(x + t\xi,\ y + t\eta,\ p + t\pi,\ r + t\rho),$$

and t is so small that its square may be neglected,

$$(a + cy)\frac{\partial}{\partial x} + (b - cx)\frac{\partial}{\partial y} - c(1 + p^2)\frac{\partial}{\partial p} - 3cpr\frac{\partial}{\partial r}$$

must annihilate f.

As the constants are independent we infer that

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + (1 + p^2)\frac{\partial}{\partial p} + 3pr\frac{\partial}{\partial r}$$

must each separately annihilate f.

We conclude therefore that in f neither x nor y can occur explicitly, so that f is a function of p and r annihilated by

$$(1 + p^2)\frac{\partial}{\partial p} + 3pr\frac{\partial}{\partial r};$$

it is now at once seen that the required differential invariant for the group of movements in the plane must be a function of $\frac{(1+p^2)^{\frac{3}{2}}}{r}$, that is, of the radius of curvature.

§ 32. In the theory of differential invariants we look on the group as known and deduce its invariants; a related problem is: 'given a differential equation or differential expression to find the infinitesimal transformations which the equation or the expression admits.'

We know that these transformations must generate a group, though we do not know that the group will be finite. It should be noticed, however, that the property of admitting an infinitesimal transformation at all belongs only to particular types of differential equations.

Thus if we take the equation

$$\frac{d^2y}{dx^2} = x^2 + y^2$$

and try whether it admits the infinitesimal transformation

$$x' = x + t\xi, \quad y' = y + t\eta, \quad p' = p + t\pi, \quad r' = r + t\rho,$$

we see that it cannot admit it unless

$$\rho = 2x\xi + 2y\eta,$$

for all values of x, y, p, r satisfying the equation $r = x^2 + y^2$.

We must therefore have

$$\frac{\partial^2\eta}{\partial x^2} + \left(2\frac{\partial^2\eta}{\partial x\,\partial y} - \frac{\partial^2\xi}{\partial x^2}\right)p + \left(\frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\,\partial y}\right)p^2 - \frac{\partial^2\xi}{\partial y^2}p^3$$
$$+ \left(\frac{\partial\eta}{\partial y} - 2\frac{\partial\xi}{\partial x} - 3p\frac{\partial\xi}{\partial y}\right)(x^2+y^2) - 2x\xi - 2y\eta = 0$$

for all values of x, y, and p.

Equating the coefficients of the different powers of p to zero, we get

$$(1)\quad \frac{\partial^2\xi}{\partial y^2} = 0, \qquad (2)\quad \frac{\partial^2\eta}{\partial y^2} - 2\frac{\partial^2\xi}{\partial x\,\partial y} = 0,$$

$$(3)\quad 2\frac{\partial^2\eta}{\partial x\,\partial y} - \frac{\partial^2\xi}{\partial x^2} - 3\frac{\partial\xi}{\partial y}(x^2+y^2) = 0,$$

$$(4)\quad \frac{\partial^2\eta}{\partial x^2} + \left(\frac{\partial\eta}{\partial y} - 2\frac{\partial\xi}{\partial x}\right)(x^2+y^2) - 2x\xi - 2y\eta = 0.$$

From (1) we see that

$$\xi = yf(x) + \phi(x);$$

by differentiating (2) with respect to x, and (3) with respect to y, and eliminating η we get

$$\frac{\partial^3 \xi}{\partial x^2 \partial y} + 2y\frac{\partial \xi}{\partial y} = 0;$$

that is

$$f''(x) + 2yf(x) = 0,$$

so that $f(x)$ vanishes identically.

From (1), (2), and (3) we therefore conclude that

$$\xi = \phi(x), \quad \eta = yf(x) + \psi(x),$$

and

$$2f'(x) = \phi''(x).$$

From (4) we get

$$yf''(x) + \psi''(x) + (x^2 + y^2)\big(f(x) - 2\phi'(x)\big) = 2x\phi(x) + 2y^2 f(x) + 2y\psi(x),$$

and on equating the coefficients of y^2 in this equation we see that

$$f(x) + 2\phi'(x) = 0,$$

and we conclude that $f'(x) = \phi''(x) = 0$.

By equating the coefficients of y we get $\psi(x) = 0$; while by equating the terms independent of y on each side we easily obtain $\phi(x) = 0$, and therefore $f(x) = 0$.

The equation proposed therefore does not admit any infinitesimal transformation.

If we were to treat the equation $\frac{d^2 y}{dx^2} = 0$ in the same manner, we should find that the only infinitesimal transformations it admits are those of the projective group.

Example. Find the form of the infinitesimal transformations which have the property of transforming any pair of curves, cutting orthogonally, into another such pair.

Let

$$x' = x + t\xi, \quad y' = y + t\eta, \quad p' = p + t\pi,$$

be the once extended infinitesimal point transformation; and let x, y be the point of intersection of the two curves, and p and q the tangents of the respective inclinations of the axis of x to the curves at this point, so that $pq + 1 = 0$.

We have now to find the form of ξ and η in order that $pq + 1 = 0$ may admit the infinitesimal transformation.

We must have

$$p\,(\eta_1+(\eta_2-\xi_1)\,q-\xi_2 q^2)+q\,(\eta_1+(\eta_2-\xi_1)\,p-\xi_2 p^2)=0$$

wherever $pq+1=0$. In this and other like examples we shall employ the suffix 1 to denote partial differentiation with respect to x, and the suffix 2 to denote partial differentiation with respect to y.

Substituting $\frac{1}{p}$ for q in this equation, and equating the different powers of p to zero, we get

$$\eta_1+\xi_2=0, \quad \xi_1-\eta_2=0,$$

so that ξ and η are conjugate functions of x and y.

An infinity of independent infinitesimal transformations will then have the required property.

§ 33. We know that the differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=0$$

is unaltered by any transformation of the group of movements of a rigid body in space; and we also know that it is unaltered by inversion with respect to any sphere; and finally that it is unaltered by the transformation

$$x'=kx, \quad y'=ky, \quad z'=kz,$$

where k is any constant, that is, by uniform expansion with respect to the origin. We therefore see that this differential equation admits a group, and we now proceed to find all the infinitesimal transformations of this group.

It is a matter of interest to connect this problem with another one, apparently different, but really the same.

Any curve in space, the tangent to which at each point on it intersects the absolute circle at infinity, is called a *minimum curve.* If x, y, z and $x+dx$, $y+dy$, $z+dz$ are two consecutive points on such a curve,

$$dx^2+dy^2+dz^2=0.$$

Through any point P in space an infinity of minimum curves can be drawn, and the tangents at P to these curves form a cone; also through P an infinity of surfaces can be drawn to satisfy the equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=0,$$

and the tangent planes to these also touch a cone; we shall now prove that these cones coincide.

On any surface, and through any point on it, two minimum curves can be drawn; for in the usual notation we have on any surface

$$dx^2+dy^2+dz^2 = dx^2+dy^2+(pdx+qdy)^2;$$

if therefore we choose $dx : dy$ so that

$$(1+p^2)\,dx^2+2\,pqdxdy+(1+q^2)\,dy^2=0,$$

we have two directions for minimum curves through the point.

Now on any surface, u = constant, which satisfies

$$(1)\quad \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=0,$$

we must have $$1+p^2+q^2=0,$$

and therefore the minimum lines on the surface drawn through any point on the surface must coincide; and, conversely, surfaces with this property satisfy the differential equation (1).

It follows that any tangent plane, at a given point, to a surface satisfying the equation (1) touches the cone, formed by the tangents to the minimum curves through the same point; the two cones therefore coincide at every point of space, and the same set of transformations must leave unaltered the two equations,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=0 \quad\text{and}\quad dx^2+dy^2+dz^2=0.$$

This is a particular case of a theorem, to be considered later, connecting partial differential equations of the first order with equations of the form

$$f(x_1,\ldots,x_n,\ dx_1,\ldots,dx_n)=0,$$

where $dx_1,\ dx_2,\ \ldots,\ dx_n$ enter the equation homogeneously. These equations are called Mongian equations.

§ 34. Consider the infinitesimal transformation

$$x'=x+t\xi,\quad y'=y+t\eta,\quad z'=z+t\zeta,$$

which has the property of being admitted by the equation

$$dx^2+dy^2+dz^2=0.$$

Since $dx'^2+dy'^2+dz'^2=0$, wherever $dx^2+dy^2+dz^2=0$,

we say that these two equations are connected; we now have the equation

$$dx(\xi_1 dx + \xi_2 dy + \xi_3 dz) + dy(\eta_1 dx + \eta_2 dy + \eta_3 dz) + dz(\zeta_1 dx + \zeta_2 dy + \zeta_3 dz) = 0$$

connected with $\quad dx^2 + dy^2 + dz^2 = 0.$

We must therefore have

$$(1) \quad \xi_1 = \eta_2 = \zeta_3, \quad \eta_3 + \zeta_2 = \zeta_1 + \xi_3 = \xi_2 + \eta_1 = 0.$$

To verify that we obtain these same equations by the condition that the two equations

$$\left(\frac{\partial u}{\partial x'}\right)^2 + \left(\frac{\partial u}{\partial y'}\right)^2 + \left(\frac{\partial u}{\partial z'}\right)^2 = 0, \text{ and } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

are connected, we write down the identities

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + t\left(\xi_1 \frac{\partial}{\partial x'} + \eta_1 \frac{\partial}{\partial y'} + \zeta_1 \frac{\partial}{\partial z'}\right),$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} + t\left(\xi_2 \frac{\partial}{\partial x'} + \eta_2 \frac{\partial}{\partial y'} + \zeta_2 \frac{\partial}{\partial z'}\right),$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} + t\left(\xi_3 \frac{\partial}{\partial x'} + \eta_3 \frac{\partial}{\partial y'} + \zeta_3 \frac{\partial}{\partial z'}\right);$$

and, since t is so small that its square may be neglected, we deduce from these

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} - t\left(\xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial z}\right),$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} - t\left(\xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y} + \zeta_2 \frac{\partial}{\partial z}\right),$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} - t\left(\xi_3 \frac{\partial}{\partial x} + \eta_3 \frac{\partial}{\partial y} + \zeta_3 \frac{\partial}{\partial z}\right).$$

By the conditions of the problem the expression

$$\frac{\partial u}{\partial x}\left(\xi_1 \frac{\partial u}{\partial x} + \eta_1 \frac{\partial u}{\partial y} + \zeta_1 \frac{\partial u}{\partial z}\right) + \frac{\partial u}{\partial y}\left(\xi_2 \frac{\partial u}{\partial x} + \eta_2 \frac{\partial u}{\partial y} + \zeta_2 \frac{\partial u}{\partial z}\right) + \frac{\partial u}{\partial z}\left(\xi_3 \frac{\partial u}{\partial x} + \eta_3 \frac{\partial u}{\partial y} + \zeta_3 \frac{\partial u}{\partial z}\right)$$

must therefore be zero, wherever the expression

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

is zero, and the equations (1) are thus obtained over again.

§ 35. We now take

$$\xi_1 = \eta_2 = \zeta_3 = f(x, y, z),$$
$$\eta_3 + \zeta_2 = \zeta_1 + \xi_3 = \xi_2 + \eta_1 = 0.$$

Differentiating $\eta_3 + \zeta_2 = 0$ with respect to y and z, and expressing the resulting equation in terms of f, we get

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Similarly we obtain

$$\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial x^2} = 0, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

and conclude that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 0.$$

We therefore take

$$f = a_0 + a_1 x + a_2 y + a_3 z + a_{23} yz + a_{31} zx + a_{12} xy + a_{123} xyz,$$

where the coefficients of the powers and products of the variables are constants, so that

$$\frac{\partial^3 \xi}{\partial x \, \partial y \, \partial z} = a_{23} + a_{123} x, \quad \frac{\partial^3 \eta}{\partial x \, \partial y \, \partial z} = a_{31} + a_{123} y,$$
$$\frac{\partial^3 \zeta}{\partial x \, \partial y \, \partial z} = a_{12} + a_{123} z.$$

By differentiating $\eta_3 + \zeta_2$ with respect to x, $\zeta_1 + \xi_3$ with respect to y, and $\xi_2 + \eta_1$ with respect to z, we have

$$\xi_{23} = \eta_{31} = \zeta_{12} = 0;$$

and conclude that

$$a_{23} = a_{31} = a_{12} = a_{123} = 0.$$

Integrating $\quad \xi_1 = f = a_0 + a_1 x + a_2 y + a_3 z$

we see that

$$\xi = a_0 x + \tfrac{1}{2} a_1 x^2 + a_2 xy + a_3 xz + F(y, z);$$

and since $\xi_{23} = 0$ we see that $F(y, z)$ must be of the form $F_{12}(y) + F_{13}(z)$, where $F_{12}(y)$ is some unknown function of y, and $F_{13}(z)$ some unknown function of z.

We have now advanced so far that we may take

$$\xi = a_0 x + \tfrac{1}{2} a_1 x^2 + a_2 xy + a_3 xz + F_{12}(y) + F_{13}(z),$$
$$\eta = a_0 y + a_1 xy + \tfrac{1}{2} a_2 y^2 + a_3 yz + F_{21}(x) + F_{23}(z),$$
$$\zeta = a_0 z + a_1 xz + a_2 yz + \tfrac{1}{2} a_3 z^2 + F_{31}(x) + F_{32}(y);$$

and from the equations

$$\eta_3+\zeta_2=\zeta_1+\xi_3=\xi_2+\eta_1=0$$

we next obtain

$$\begin{aligned} a_3 y+F'_{23}(z)+a_2 z+F'_{32}(y)&=0,\\ a_1 z+F'_{31}(x)+a_3 x+F'_{13}(z)&=0,\\ a_2 x+F'_{12}(y)+a_1 y+F'_{21}(x)&=0. \end{aligned}$$

We conclude then that

$$\begin{aligned} F_{32}(y)&=-\tfrac{1}{2}a_3 y^2-A_1 y+\text{constant},\\ F_{23}(z)&=-\tfrac{1}{2}a_2 z^2+A_1 z+\text{constant}, \end{aligned}$$

with similar expressions for the other functions.

Finally we have

$$\begin{aligned} \xi&=\tfrac{1}{2}a_1(x^2-y^2-z^2)+a_2 yx+a_3 xz+a_0 x+a+A_2 z-A_3 y,\\ \eta&=\tfrac{1}{2}a_2(y^2-z^2-x^2)+a_3 yz+a_1 xy+a_0 y+\beta+A_3 x-A_1 z,\\ \zeta&=\tfrac{1}{2}a_3(z^2-x^2-y^2)+a_1 xz+a_2 yz+a_0 z+\gamma+A_1 y-A_2 x. \end{aligned}$$

We now have ten infinitesimal transformations admitted by the equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=0,$$

and by the Mongian equation

$$dx^2+dy^2+dz^2=0.$$

The ten operators which correspond to these transformations are

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z},\quad y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z},\quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},$$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\quad (y^2+z^2-x^2)\frac{\partial}{\partial x}-2xy\frac{\partial}{\partial y}-2zx\frac{\partial}{\partial z},$$

$$(z^2+x^2-y^2)\frac{\partial}{\partial y}-2xy\frac{\partial}{\partial x}-2yz\frac{\partial}{\partial z},$$

$$(x^2+y^2-z^2)\frac{\partial}{\partial z}-2yz\frac{\partial}{\partial y}-2zx\frac{\partial}{\partial z}.$$

§ 36. *Example.* Find the most general infinitesimal transformation with the property of transforming any two surfaces intersecting orthogonally into another pair of such surfaces.

Let u and v be any two functions satisfying the equation

$$(1)\qquad \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}=0,$$

then $u=\text{constant}$, and $v=\text{constant}$ will be two surfaces intersecting orthogonally.

The equation (1) must therefore admit

$$x' = x + t\xi, \quad y' = y + t\eta, \quad z' = z + t\zeta.$$

We have

$$\frac{\partial u}{\partial x'} = \frac{\partial u}{\partial x} - t\left(\xi_1 \frac{\partial u}{\partial x} + \eta_1 \frac{\partial u}{\partial y} + \zeta_1 \frac{\partial u}{\partial z}\right),$$

with similar expressions for

$$\frac{\partial u}{\partial y'}, \quad \frac{\partial u}{\partial z'}, \quad \frac{\partial v}{\partial x'}, \quad \frac{\partial v}{\partial y'}, \quad \frac{\partial v}{\partial z'};$$

substituting in (1) and neglecting t^2 we see that

$$\begin{aligned} &2\xi_1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\eta_2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + 2\zeta_3 \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} \\ &+ (\eta_3 + \zeta_2)\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right) + (\xi_3 + \zeta_1)\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right) \\ &\qquad + (\xi_2 + \eta_1)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = 0 \end{aligned}$$

is an equation connected with (1).

We are thus again led to the equations

$$\xi_1 = \eta_2 = \zeta_3, \quad \eta_3 + \zeta_2 = \xi_3 + \zeta_1 = \xi_2 + \eta_1 = 0;$$

and conclude that the only infinitesimal transformations with the required property are those found in the last article.

CHAPTER III

THE GENERATION OF A GROUP FROM ITS INFINITESIMAL TRANSFORMATIONS

The identical transformation.

§ 37. From the equations

$$x'_i = f_i(x, a), \qquad (i = 1, \ldots, n)$$

which define a group, and from

$$x''_i = f_i(x', b) = f_i(x, c),$$

we have

$$(1) \qquad c_k = \phi_k(a, b), \qquad (k = 1, \ldots, r).$$

Subject to certain limitations on the values of $a_1, \ldots, a_r$, $c_1, \ldots c_r$, we can deduce from these equations

$$(2) \qquad b_k = \psi_k(a, c), \qquad (k = 1, \ldots, r).$$

Now suppose that on taking $a_1 = c_1, \ldots, a_r = c_r$ the functions $\psi_k(a, c)$ remain analytic functions of their arguments; and suppose further that the values of $b_1, \ldots, b_r$ so obtained make $f_i(x'_1, \ldots, x'_n, b_1, \ldots, b_r)$ an analytic function of its arguments, within the region over which $x'_1, \ldots, x'_n$ may range; then as we have always

$$f_i(x, c) = f_i(x', b);$$

by the hypothesis $a_k = c_k$ we have

$$x'_i = f_i(x, a) = f_i(x, c),$$

so that $\qquad x'_i = f_i(x', b), \qquad (i = 1, \ldots, n):$

that is, $b_k = \psi_k(a, a)$ gives the identical transformation.

Since these values of $b_1, \ldots, b_r$ are obtained from the equations

$$a_k = \phi_k (a_1, \ldots, a_r, b_1, \ldots, b_r),$$

it might seem at first as if they would be functions of $a_1, \ldots, a_r$: this, however, is not the case; they are absolutely independent of $a_1, \ldots, a_r$. To prove this, suppose that

$$b_k = \lambda_k (a_1, \ldots, a_r), \qquad (k = 1, \ldots, r),$$

λ_k being some functional symbol: then

$$x'_i = f_i (x'_1, \ldots, x'_n, \lambda_1, \ldots, \lambda_r),$$

and as $\lambda_1, \ldots, \lambda_r$ must occur effectively in f_i we should have x'_i expressed in terms of $x'_1, \ldots, x'_n$ and arbitrary constants, which is of course impossible.

§ 38. As an example in finding the parameters which give the identical transformation we take the case of the linear group

$$x'_i = \sum^{h=n} a_{hi} x_h.$$

We have

$$c_{hi} = \sum^{k=n} a_{hk} b_{ki}:$$

putting $c_{hi} = a_{hi}$ we have

$$\sum^{k=n} a_{hk} b_{ki} = a_{hi},$$

and therefore, since the determinant

$$\begin{vmatrix} a_{11}, & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix}$$

cannot be zero, we must have $b_{hi} = 0$, if h and i are unequal, and $b_{ii} = 1$.

Of course these values of the parameters for the identical transformation could have been obtained by inspection of the equations of the group, but we have preferred to deduce them by the general method in order to illustrate the theorem that they are absolute constants.

As we shall very often have to deal with constants such as b_{hi}, characterized by the property of being zero if h and i are unequal, and unity if they are equal, it will be convenient to denote such a constant always by the symbol ϵ_{hi}.

We should thus express the parameters of the identical transformation in the general linear group by the equations

$$b_{hi} = \epsilon_{hi}, \qquad \binom{h = 1, \ldots, n}{i = 1, \ldots, n},$$

but it will not always be necessary to explicitly state the range of the suffixes.

§ 39. Engel has proved that finite continuous groups do not necessarily contain the identical transformation.

Thus consider the function due to Poincaré

$$y = \sum_{n=0}^{n=\infty} 2^{-n} x^{3^n},$$

which is known (Forsyth, *Theory of Functions*, § 87, Ex. 3) to exist only within a circle of radius unity, whose centre is the origin. It follows that x is an analytic function of y such that, whatever value y takes, x always lies within a circle of radius unity. Let $x = \lambda(y)$: then λ is a function such that, whatever may be the value of its argument, it is always less than unity.

Take now the transformation schemes $x' = \lambda(a)\, x$. These clearly generate a group; for if

$$x'' = \lambda(b)\, x' \quad \text{then} \quad x'' = \lambda(a)\, \lambda(b)\, x,$$

and $\lambda(a)\lambda(b) = k$, k being a constant less than unity, so that $\lambda(a)\lambda(b) = \lambda(c)$, where

$$c = \sum_{n=0}^{n=\infty} 2^{-n} k^{3^n}.$$

We therefore have the group property, since we can deduce from $x' = \lambda(a)\, x$ and $x'' = \lambda(b) x'$ the equation $x'' = \lambda(c) x$.

We now have $$\lambda(b) = \frac{\lambda(c)}{\lambda(a)},$$

but we cannot take $c = a$, for that would give $\lambda(b) = 1$, which is impossible, since $\lambda(b)$ is always less than unity.

The method of obtaining the operators of a group.

§ 40. Let (1) $x_i' = f_i(x, a)$

be a transformation of the group; let $\dfrac{\partial x_i'}{\partial a_k}$, expressed in terms

of $x'_1, \dots, x'_n$, $a_1, \dots, a_r$ be written $\xi_{ki}(x'_1, \dots, x'_n, a_1, \dots, a_r)$, or in abridged notation ${}_a\xi'_{ki}$; and denote by ${}_aX'_k$ the linear operator

$$ {}_a\xi'_{k1}\frac{\partial}{\partial x'_1} + \dots + {}_a\xi'_{kn}\frac{\partial}{\partial x'_n}. $$

Let $\dfrac{d}{da_k}$ denote the operation of differentiating totally with respect to a_k any function of $x'_1, \dots, x'_n$, $a_1, \dots, a_r$, in which on account of (1) $x'_1, \dots, x'_n$ are to be considered implicit functions of $a_1, \dots, a_r$.

We have

$$ \frac{d}{da_k}\phi(x'_1, \dots, x'_n, a_1, \dots, a_r) = \frac{\partial x'_1}{\partial a_k}\frac{\partial \phi}{\partial x'_1} + \dots + \frac{\partial x'_n}{\partial a_k}\frac{\partial \phi}{\partial x'_n} + \frac{\partial \phi}{\partial a_k} $$
$$ = \left({}_a\xi'_{k1}\frac{\partial}{\partial x'_1} + \dots + {}_a\xi'_{kn}\frac{\partial}{\partial x'_n} + \frac{\partial}{\partial a_k}\right)\phi; $$

that is, if we express any function of $x'_1, \dots, x'_n$, $a_1, \dots, a_r$, in terms of $x_1, \dots, x_n$, $a_1, \dots, a_r$ by means of the equation system (1), and then differentiate with respect to a_k, we get the same result as if we had performed the operation

$$ {}_aX'_k + \frac{\partial}{\partial a_k} $$

directly on the given function.

If we now keep $x_1, \dots, x_n$, $a_1, \dots, a_r$ fixed, $x'_1, \dots, x'_n$ will also remain fixed; and the increment of any function $\phi(x''_1, \dots, x''_n)$,

where $$ x''_i = f_i(x', b) = f_i(x, c), $$

due to the increment db_k, (the other parameters $b_1, \dots, b_{k-1}$, $b_{k+1}, \dots, b_r$ remaining fixed), will be

$$ {}_bX''_k\phi(x''_1, \dots, x''_n)\,db_k. $$

Since, however, $x''_i = f_i(x, c)$ and $x_1, \dots, x_n$ remain fixed, while $c_1, \dots, c_r$ are functions of $a_1, \dots, a_r$, $b_1, \dots, b_r$, we may write this increment in the form

$$ \sum^{j=r}\frac{\partial c_j}{\partial b_k}\,{}_cX''_j\,\phi(x''_1, \dots, x''_n)\,db_k. $$

Now $\phi(x''_1, \dots, x''_n)$ is an *arbitrary* function of its arguments; so that we obtain the identity

$$ {}_bX''_k = \sum^{j=r}\frac{\partial c_j}{\partial b_k}\,{}_cX''_j $$

by equating the above two expressions for the increment.

By giving k the values $1, \dots, r$ we have r identities which hold for all values of $x_1'', \dots, x_n''$, $a_1, \dots, a_r$, $b_1, \dots, b_r$, where

$$c_k = \phi_k(a, b), \qquad (k = 1, \dots, r).$$

§ 41. We now take $b_1, \dots, b_r$ to be the parameters of the identical transformation, and since these are absolute constants, we shall omit the b in ${}_bX_k''$ and write it X_k'' simply.

$\dfrac{\partial c_j}{\partial b_k}$ is now a function of $a_1, \dots, a_r$ only, for $b_1, \dots, b_r$ are absolute constants; we write it therefore in the form $a_{kj}(a_1, \dots, a_n)$, or simply a_{kj}.

Also, since $b_1, \dots, b_r$ are the parameters of the identical transformation, $c_k = a_k$, and we have the identities

$$(1) \qquad \begin{array}{l} X_1 \equiv a_{11}\ {}_aX_1 + \dots + a_{1r}\ {}_aX_r, \\ \vdots \\ X_r \equiv a_{r1}\ {}_aX_1 + \dots + a_{rr}\ {}_aX_r, \end{array}$$

where the determinant

$$\begin{vmatrix} a_{11}, & \cdot & \cdot & \cdot & a_{1r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r1}, & \cdot & \cdot & \cdot & a_{rr} \end{vmatrix}$$

cannot vanish identically, that being a condition for the existence of an identical transformation.

From these identities we deduce

$$(2) \qquad \begin{array}{l} {}_aX_1 \equiv \lambda_{11}\ X_1 + \dots + \lambda_{1r}\ X_r, \\ \vdots \\ {}_aX_r \equiv \lambda_{r1}\ X_1 + \dots + \lambda_{rr}\ X_r, \end{array}$$

where $\lambda_{ij}, \dots$ are functions of $a_1, \dots, a_r$; that is, *any operator with any implicit set of constants* $a_1, \dots, a_r$ *is dependent on* $X_1, \dots, X_r$.

This theorem is called the first fundamental theorem in group theory.

§ 42. *A group of order r contains exactly r independent operators.*

Lemma. If we have any linear operator of the form

$$(1) \qquad \sum^{k=r} a_k \frac{\partial}{\partial a_k},$$

where a_k is a function of $a_1, \ldots, a_r$, we know from the theory of differential equations that there are exactly $(r-1)$ functions of $a_1, \ldots, a_r$ which this operator will annihilate. Let $A_1, \ldots, A_{r-1}$ be any such $(r-1)$ functionally unconnected functions, then if f is any function of $a_1, \ldots, a_r$, which is annihilated by (1), we know that it must be a function of $A_1, \ldots, A_{r-1}$.

It follows that there cannot be any linear operator of the form (1) which annihilates the n functions $f_1, \ldots, f_n$ defining a group; for if there were such an operator there could not be more than $(r-1)$ effective constants involved in $f_1, \ldots, f_n$, viz. $A_1, \ldots, A_{r-1}$.

From this lemma we conclude that there cannot be any equation system of the form

$$\sum^{k=r} \lambda_k \frac{\partial x_i'}{\partial a_k} = 0, \qquad (i = 1, \ldots, n),$$

where $\lambda_1, \ldots, \lambda_n$ do not contain $x_1, \ldots, x_n$; and therefore there cannot be any identical relation of the form

$$\sum^{k=r} \lambda_k \; {}_aX_k = 0$$

between the operators ${}_aX_1, \ldots, {}_aX_r$ when $\lambda_1, \ldots, \lambda_r$ only involve $a_1, \ldots, a_r$; that is, the r operators

$${}_aX_1, \ldots, {}_aX_r$$

are independent, and therefore so are the operators

$$X_1, \ldots, X_r.$$

If $b_1, \ldots, b_r$ are the parameters of the identical transformation, and $b_1 + e_1, \ldots, b_r + e_r$ an adjacent set of parameters, $e_1, \ldots, e_r$ being so small that their squares may be neglected, then expanding $\quad x_i' = f_i(x_1, \ldots, x_n, \; b_1 + e_1, \ldots, b_r + e_r)$

by Taylor's theorem we have

$$x_i' = x_i + \sum^{k=r} e_k \, \xi_{ki}', \qquad (i = 1, \ldots, n);$$

or since x_i' is approximately equal to x_i,

$$x_i' = x_i + \sum^{k=r} e_k \, \xi_{ki}.$$

Since $$X_k = \sum^{i=n} \xi_{ki} \frac{\partial}{\partial x_i},$$

and the operators are independent, we see that there are

exactly r independent infinitesimal transformations; and we see further that the operators of a group, as defined in § 13, coincide with the operators as defined in this chapter.

§ 43. As an example illustrative of the preceding methods we take the projective group of space, viz.

$$(1) \quad x'_i = \frac{a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3 + a_{4i}}{a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}}, \qquad (i = 1, 2, 3);$$

from these equations we obtain (p being < 4)

$$\frac{\partial x'_i}{\partial a_{pq}} = \frac{x_p}{a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}} \quad \text{if} \quad q < 4,$$

and

$$\frac{\partial x'_i}{\partial a_{p4}} = -\frac{a_{1i}x_1x_p + a_{2i}x_2x_p + a_{3i}x_3x_p + a_{4i}x_p}{(a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44})^2}.$$

If A_{pq} is the minor of a_{pq} in the determinant

$$M = \begin{vmatrix} a_{11}, & . & . & . & a_{14} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{41}, & . & . & . & a_{44} \end{vmatrix},$$

we have, as the scheme inverse to (1),

$$x_i = \frac{A_{i1}x'_1 + A_{i2}x'_2 + A_{i3}x'_3 + A_{i4}}{A_{41}x'_1 + A_{42}x'_2 + A_{43}x'_3 + A_{44}}.$$

Since only the ratios of the constants are involved, we may take a_{44} as absolutely fixed; and we get as the operator corresponding to a_{pq}

$$(2) \quad M^{-1}(A_{p1}x'_1 + A_{p2}x'_2 + A_{p3}x'_3 + A_{p4})\frac{\partial}{\partial x'_q} \quad \text{if } q < 4.$$

If $q = 4$ the operator is

(3)

$$M^{-1}(A_{p1}x'_1 + A_{p2}x'_2 + A_{p3}x'_3 + A_{p4})\left(x'_1\frac{\partial}{\partial x'_1} + x'_2\frac{\partial}{\partial x'_2} + x'_3\frac{\partial}{\partial x'_3}\right).$$

The identical transformation is obtained by taking $a_{pq} = \epsilon_{pq}$: this gives $A_{pq} = \epsilon_{pq}$, and the corresponding 15 operators are

$$(4) \quad \begin{cases} x'_p\dfrac{\partial}{\partial x'_q}, & \begin{pmatrix} p = 1, 2, 3 \\ q = 1, 2, 3 \end{pmatrix}, \\ \dfrac{\partial}{\partial x'_p}, & (p = 1, 2, 3), \\ x'_p\left(x'_1\dfrac{\partial}{\partial x'_1} + x'_2\dfrac{\partial}{\partial x'_2} + x'_3\dfrac{\partial}{\partial x'_3}\right), & (p = 1, 2, 3). \end{cases}$$

The reader may easily verify that the set of 15 operators given by (2) and (3) is dependent on the set of 15 given by (4); and also that either of these sets of operators contains 15 independent operators.

Examples. Find the infinitesimal operators of

(1) the projective group of the plane;

(2) the orthogonal linear homogeneous group, viz.

$$x' = a_{11}x + a_{21}y + a_{31}z, \quad y' = a_{21}x + a_{22}y + a_{32}z,$$
$$z' = a_{31}x + a_{32}y + a_{33}z,$$

where the constants are such that

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2;$$

(3) the linear homogeneous group in n variables;

(4) the non-projective group given in § 25.

The canonical equations of a group.

§ 44. The parameters $b_1, \ldots, b_r$ which determine the identical transformation in the group

$$x_2' = f_i(x, a)$$

give for all values of the parameters $a_1, \ldots, a_r$

$$a_k = \phi_k(a_1, \ldots, a_r, \; b_1, \ldots, b_r);$$

they are therefore the same parameters as those which determine the identical transformation in the first parameter group (cf. (3), § 19).

It also follows from the definition of the functions

$$a_{kj}(a_1, \ldots, a_r)$$

that the infinitesimal operators of the first parameter group are $A_1, \ldots, A_r$ where

$$A_k \equiv a_{k1}\frac{\partial}{\partial a_1} + \ldots + a_{kr}\frac{\partial}{\partial a_r}, \qquad (k = 1, \ldots, r).$$

Let now $a_1^0, \ldots, a_r^0$ be the *initial* values of the variables $a_1, \ldots, a_r$; let the operator

$$e_1 A_1 + \ldots + e_r A_r$$

be written A; and the operator obtained by replacing $a_1, \ldots, a_r$ in A by $a_1^0, \ldots, a_r^0$ respectively be written A_0.

If X is any linear operator, we shall denote by e^X the expression

$$1 + \frac{1}{1!}X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \ldots \text{ to infinity.}$$

We now take

$$a_k = e^{tA_0} a_k^0, \qquad (k = 1, \ldots, r),$$

when we have
$$\frac{d}{dt} a_k = A_0 a_k,$$

and therefore, $\phi(a_1, \ldots, a_r)$ being any function of $a_1, \ldots, a_r$,

$$\frac{d}{dt} \phi(a_1, \ldots, a_r) = A_0 \phi(a_1, \ldots, a_r).$$

We also have
$$\frac{d}{dt} A_0 = A_0 \frac{d}{dt}$$

since the operators are in unconnected sets of variables, viz. t and $a_1^0, \ldots, a_r^0$; and therefore

$$\frac{d^2}{dt^2} \phi = \frac{d}{dt} A_0 \phi = A_0 \frac{d}{dt} \phi = A_0^2 \phi.$$

Similarly we have

$$\frac{d^i}{dt^i} \phi = A_0{}^i \phi,$$

and therefore the limit of $\dfrac{d^i \phi}{dt^i}$, when t is zero, is

$$A_0{}^i \phi (a_1^0, \ldots, a_r^0).$$

Since $\phi(a_1, \ldots, a_r)$ is a function of t and of the initial values $a_1^0, \ldots, a_r^0$, we have by Taylor's theorem

$$\phi(a_1, \ldots, a_r) = \phi_{t=0} + t \left(\frac{d\phi}{dt}\right)_{t=0} + \frac{t^2}{2\,!} \left(\frac{d^2\phi}{dt^2}\right)_{t=0} + \ldots;$$

and therefore

$$\phi(a_1, \ldots, a_r) = \left(1 + \frac{t}{1\,!} A_0 + \frac{t^2}{2\,!} A_0^2 + \ldots\right) \phi(a_1^0, \ldots, a_r^0).$$

From this formula we deduce

$$\begin{aligned} \frac{d}{dt} \phi(a_1, \ldots, a_r) &= A_0 \left(1 + \frac{t}{1\,!} A_0 + \frac{t^2}{2\,!} A_0^2 + \ldots\right) \phi(a_1^0, \ldots, a_r^0), \\ &= \left(1 + \frac{t}{1\,!} A_0 + \frac{t^2}{2\,!} A_0^2 + \ldots\right) A_0 \phi(a_1^0, \ldots, a_r^0), \\ &= A \phi(a_1, \ldots, a_r), \end{aligned}$$

by a second application of the same formula.

A particular case of this second formula is

$$(1) \qquad \frac{da_k}{dt} = \sum^{s=r} e_s a_{sk}.$$

The identities of § 41 (expressed in the variables $x'_1, \ldots, x'_n$)

$$_aX'_k = \lambda_{k1} X'_1 + \ldots + \lambda_{kr} X'_r, \qquad (k = 1, \ldots, r)$$

are equivalent to

(2) $$_a\xi'_{ki} = \lambda_{k1} \xi'_{1i} + \ldots + \lambda_{kr} \xi'_{ri};$$

and therefore, since x'_1 is a function of $x_1, \ldots, x_n$, $a_1, \ldots, a_r$ and thus implicitly of $x_1, \ldots, x_n$, $a_1{}^0, \ldots, a_r{}^0$, t, and since

$$\frac{\partial x'_i}{\partial a_k} = {}_a\xi'_{ki},$$

we have $$\frac{dx'_i}{dt} = \sum^{k=j=s=r} \lambda_{kj} \xi'_{ji} e_s a_{sk}$$

by (1) and (2).

Now the identities (1) and (2) of § 41 are equivalent, so that

we must have $$\sum^{k=r} \lambda_{kj} a_{sk} = \epsilon_{sj};$$

and therefore

(3) $$\frac{dx'_i}{dt} = \sum^{s=r} e_s \xi'_{si}.$$

We can deduce from the formula (3) a result which will be useful later; since

$$x'_i = f_i(x, a), \qquad (i = 1, \ldots, n)$$

we have the inverse scheme

$$x_i = F_i(x', a);$$

and therefore, since x_i does not involve t, we see that

$$\frac{d}{dt} F_i(x'_1, \ldots, x'_n, a_1, \ldots, a_r) \equiv 0.$$

It follows from (1) and (3) that the operator

$$\sum^{s=k=r,\, j=n} e_s \left(\xi'_{sj} \frac{\partial}{\partial x'_j} + a_{sk} \frac{\partial}{\partial a_k} \right),$$

that is, the operator $$\sum^{s=r} e_s (X'_s + A_s)$$

annihilates every function of $x_1, \ldots, x_n$ when expressed in terms of $x'_1, \ldots, x'_n$, $a_1, \ldots, a_r$. If we notice that $x'_1, \ldots, x'_n$, $a_1, \ldots, a_r$, $e_1, \ldots, e_r$ are all independent of one another, we shall see that each of the operators $X'_1 + A_1, \ldots, X'_r + A_r$, must have this property.

If we now take $a_1^0, \ldots, a_r^0$ to be the parameters of the identical transformation, then, when $t = 0$, $x_i' = x_i$; and applying Taylor's theorem we have

$$x_i' = x_i + t\left(\frac{dx_i'}{dt}\right)_{t=0} + \frac{t^2}{2!}\left(\frac{d^2 x_i'}{dt^2}\right)_{t=0} + \ldots.$$

If we write X' for the linear operator

$$e_1 X_1' + \ldots + e_r X_r',$$

and express any function of $x_1', \ldots, x_n'$ in terms of $x_1, \ldots, x_n, t$, $e_1, \ldots, e_r$ we have from (3)

$$\frac{d}{dt}\phi(x_1', \ldots, x_n') = X'\phi(x_1', \ldots, x_n').$$

Now $X'\phi(x_1', \ldots, x_n')$ is itself a function of $x_1', \ldots, x_n'$, so that

$$\frac{d}{dt}X'\phi(x_1', \ldots, x_n') = X'^2\phi(x_1', \ldots, x_n'),$$

and therefore

$$\frac{d^2}{dt^2}\phi(x_1', \ldots, x_n') = X'^2\phi(x_1', \ldots, x_n'),$$

and more generally

$$\frac{d^m}{dt^m}\phi(x_1', \ldots, x_n') = X'^m\phi(x_1', \ldots, x_n').$$

It follows that the limit of $\left(\frac{d^m x_i'}{dt^m}\right)_{t=0}$ is $X^m x_i$, and therefore

$$x_i' = \left(1 + \frac{t}{1!}X + \frac{t^2}{2!}X^2 + \ldots\right)x_i = e^{tX}x_i.$$

Similarly we could prove that

$$\text{(4)} \qquad \phi(x_1', \ldots, x_n') = e^{tX}\phi(x_1, \ldots, x_n),$$

where X denotes the operator

$$e_1 X_1 + \ldots + e_r X_r.$$

Example. Assuming that

$x_i' = e^{tX}x_i$, prove that $\phi(x_1', \ldots, x_n') = e^{tX}\phi(x_1, \ldots, x_n)$.

Since $A_1, \ldots, A_r$ are operators given by

$$A_k \equiv a_{k1}\frac{\partial}{\partial a_1} + \ldots + a_{kr}\frac{\partial}{\partial a_r}, \qquad (k = 1, \ldots, r)$$

where the determinant

$$\begin{vmatrix} a_{11}, & . & . & . & a_{1r} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{r1}, & . & . & . & a_{rr} \end{vmatrix}$$

does not vanish identically, these operators are not merely *independent* but also *unconnected.*

A group in n variables with n unconnected operators is said to be *transitive*; if the order of the group is also equal to n the group is said to be *simply transitive.*

We now see that the first parameter group is simply transitive.

Since $A_1, \ldots, A_r$ are unconnected operators, and $e_1, \ldots, e_r$ arbitrary parameters, and $a_1, \ldots, a_r$ are defined by

$$a_k = e^{t(A_0)} a_k{}^0, \qquad (k = 1, \ldots, r),$$

we know that there can be no functional connexion between $a_1, \ldots, a_r$, they may therefore be any parameters whatever.

It follows that if

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n),$$

we can always throw $f_i(x, a)$ into the form

$$e^{e_1 X_1 + \ldots + e_r X_r} x_i.$$

When the equations of a group are given in the form

$$x_i' = e^{e_1 X_1 + \ldots + e_r X_r} x_i, \qquad (i = 1, \ldots, n),$$

the group is said to be in *canonical form.*

Since $e^{e_1 X_1 + \ldots + e_r X_r}$ is the limit when $m = \infty$ of

$$\left(1 + \frac{e_1 X_1 + \ldots + e_r X_r}{m}\right)^m,$$

we see *that every finite operation of a group can be generated by indefinite repetition of an infinitesimal operation.*

It should be noticed that the operation of substituting for $x_1, \ldots, x_n$ in any given function of these variables $x_1', \ldots, x_n'$ respectively, an operation denoted in the first chapter of this treatise by $S a_1, \ldots, a_r$, has now been proved equivalent to operating on $x_1, \ldots, x_n$ with $e^{e_1 X_1 + \ldots + e_r X_r}$, when $e_1, \ldots, e_r$ are functions of $a_1, \ldots, a_r$ known as the canonical parameters. We shall sometimes speak of $e^{e_1 X_1 + \ldots + e_r X_r}$ as

a finite operator of the group, or simply as an operator, when there is no risk of confusing it with a linear operator.

When in canonical form, the parameters of a transformation scheme and its inverse are very simply related.

We have seen that

$$\phi(x_1', \dots, x_n') = e^X \phi(x_1, \dots, x_n),$$

and since this formula holds for *any* function of $x_1', \dots, x_n'$ we must also have

$$e^{-X'} \phi(x_1', \dots, x_n') = e^X e^{-X} \phi(x_1, \dots, x_n).$$

Now just as in elementary algebra we see that

$$e^X e^{-X} \equiv 1,$$

and therefore $\phi(x_1, \dots, x_n) = e^{-X'} \phi(x_1', \dots, x_n')$.

A particular case of this general formula is

$$x_i = e^{-e_1 X'_1 - \dots - e_r X'_r} x_i',$$

so that the canonical parameters of any transformation scheme being $e_1, \dots, e_r$, those of the inverse scheme are $-e_1, \dots, -e_r$.

Examples. (1) Prove that, X being any linear operator,

$$x_i' = e^{tX} x_i, \qquad (i = 1, \dots, n)$$

is a group of order unity.

(2) If X and Y are two linear operators whose alternant is zero, prove that any transformation

$$x_i' = e^{tX} x_i$$

is permutable with any transformation

$$x_i' = e^{tY} x_i.$$

§ 45. When we are given the infinitesimal transformations of a group—and the group is generally discovered through the infinitesimal transformations—we are given the group in its canonical form; the question then arises, How are we to determine whether a known set of linear operators do, or do not, generate a finite continuous group?

This question will be answered in the next chapter, but just now it will be assumed that $X_1, \dots, X_r$ are r linear operators, known to generate a group given by

$$x_i' = e^{e_1 X_1 + \dots + e_r X_r} x_i, \qquad (i = 1, \dots, n).$$

The group is, however, only given in the form of an infinite series, involving the evaluation of such terms as

$$(e_1 X_1 + \dots + e_r X_r)^m x_i,$$

so that we may ask, Can $x_1', \dots, x_n'$ be expressed as finite functions of $x_1, \dots, x_n$?

The differential equation

$$(e_1 X_1 + \dots + e_r X_r)\, u = 1$$

has n unconnected integrals; let these be

$$\phi_1(x_1, \dots, x_n), \dots, \phi_n(x_1, \dots, x_n).$$

If we take as a new set of variables $y_1, \dots, y_n$ where

$$y_1 = \phi_1(x_1, \dots, x_n), \quad y_2 = \phi_2 - \phi_1, \dots, \quad y_n = \phi_n - \phi_1,$$

we see that $\qquad (e_1 X_1 + \dots + e_r X_r)\, y_1 = 1,$

and $\qquad (e_1 X_1 + \dots + e_r X_r)\, y_i = 0 \quad \text{if } i > 1;$

and therefore the operator

$$X \equiv e_1 X_1 + \dots + e_r X_r,$$

expressed in the new variables, is $\dfrac{\partial}{\partial y_1}$.

Now we have proved that $\phi(x_1, \dots, x_n)$ being any function of the variables $\phi(x_1', \dots, x_n') = e^X \phi(x_1, \dots, x_n)$, and therefore we conclude that

$$\begin{aligned} \phi_2(x_1', \dots, x_n') - \phi_1(x_i', \dots, x_n') &= \phi_2(x_1, \dots, x_n) - \phi_1(x_1, \dots, x_n) \\ &\vdots \\ \phi_n(x_1', \dots, x_n') - \phi_1(x_1', \dots, x_n') &= \phi_n(x_1, \dots, x_n) - \phi_1(x_1, \dots, x_n), \end{aligned}$$

while

$$\phi_1(x_1', \dots, x_n') = e^{\frac{\partial}{\partial y_1}} y_1 = y_1 + 1 = \phi_1(x_1, \dots, x_n) + 1.$$

From these n equations we can therefore deduce the expressions for $x_1', \dots, x_n'$ in terms of $x_1, \dots, x_n$.

It follows that, when we are given the infinitesimal operators of a group, we can find the equations of the group in finite terms if we can find the integrals $\phi_1, \dots, \phi_n$ of

$$(e_1 X_1 + \dots + e_r X_r)\, u = 1,$$

and then solve the equations

$$\phi_i(x_1', \dots, x_n') = \phi_i(x_1, \dots, x_n) + 1, \qquad (i = 1, \dots, n),$$

so as to express $x_1', \dots, x_n'$ finitely in terms of $x_1, \dots, x_n$.

The functions $\phi_1, \dots, \phi_n$ will of course involve the arbitrary parameters $e_1, \dots, e_r$.

Example. The operators

$$\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad (xy - z)\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z},$$

are known to generate a group; find the equations of the group in finite form.

We have to find the integrals of

$$e_3(xy-z)\frac{\partial u}{\partial x}+(e_1+e_2y+e_3y^2)\frac{\partial u}{\partial y}+(e_1x+e_2z+e_3yz)\frac{\partial u}{\partial z}=1.$$

The subsidiary equations are

$$\frac{dx}{e_3(xy-z)}=\frac{dy}{e_1+e_2y+e_3y^2}=\frac{dz}{e_1x+e_2z+e_3yz}=\frac{du}{1},$$

and if we write

$$a=\frac{\sqrt{4e_1e_3-e_2^2}}{2e_3},\qquad a\tan\phi=y+\frac{e_2}{2e_3},\qquad a\tan\theta=\frac{z}{x}+\frac{e_2}{2e_3},$$

these equations become

$$\frac{d\log x}{\tan\phi-\tan\theta}=d\phi=d\theta=\frac{\sqrt{4e_1e_3-e_2^2}}{2}\,du.$$

So that

$$u=\frac{2\phi}{\sqrt{4e_1e_3-e_2^2}}=\frac{2}{\sqrt{4e_1e_3-e_2^2}}\tan^{-1}\frac{(2e_3y+e_2)}{\sqrt{4e_1e_3-e_2^2}}$$

is an integral of the proposed equation; and $\dfrac{x\cos\phi}{\cos\theta}$, and $\phi-\theta$, are functions annihilated by the operator

$$e_3(xy-z)\frac{\partial}{\partial x}+(e_1+e_2y+e_3y^2)\frac{\partial}{\partial y}+(e_1x+e_2z+e_3yz)\frac{\partial}{\partial z};$$

that is $\dfrac{e_3z^2+e_2zx+e_1x^2}{e_3y^2+e_2y+e_1}$ and $\dfrac{z-xy}{2e_3yz+e_2(xy+z)+2e_1x}$

are annihilated by this operator.

The finite equations therefore of the required group are

$$\frac{e_3z'^2+e_2z'x'+e_1x'^2}{e_3y'^2+e_2y'+e_1}=\frac{e_3z^2+e_2zx+e_1x^2}{e_3y^2+e_2y+e_1},$$

$$\frac{z'-x'y'}{2e_3y'z'+e_2(x'y'+z')+2e_1x'}=\frac{z-xy}{2e_3yz+e_2(xy+z)+2e_1x},$$

$$\frac{2}{\sqrt{4e_1e_3-e_2^2}}\tan^{-1}\frac{2e_3y'+e_2}{\sqrt{4e_1e_3-e_2^2}}$$

$$=\frac{2}{\sqrt{4e_1e_3-e_2^2}}\tan^{-1}\frac{2e_3y+e_2}{\sqrt{4e_1e_3-e_2^2}}+1;$$

and if we were to solve these, and thus express x', y', z' in terms of x, y, z, we should have the finite equations of the group in canonical form.

§ 46. There is generally considerable difficulty in expressing the equations of a group in finite form when we are given the infinitesimal operators; but for most parts of the theory of groups the knowledge of the forms of the infinitesimal operators is of more interest than the knowledge of the finite form; and the most important result which we have proved in this chapter is that every transformation of a group may be obtained by indefinite repetition of a properly chosen infinitesimal transformation.

Thus if we take the binary quantic

$$u \equiv a_0 x^p + p a_1 x^{p-1} y + \ldots,$$

and apply the linear transformation

$$x' = l_1 x + m_1 y, \quad y' = l_2 x + m_2 y,$$

we get

$$u \equiv a_0' x'^p + p a_1' x'^{p-1} y' + \ldots.$$

From the identity of these two expressions for u, we deduce

$$(1) \qquad \begin{aligned} a_0' &= a_0 l_1^p + p a_1 l_1^{p-1} l_2 + \ldots, \\ a_1' &= a_0 l_1^{p-1} m_1 + \ldots, \end{aligned}$$

with similar expressions for $a_2', \ldots$; and the problem of the invariant theory is the deduction of the functions which have the property

$$f(a_0', a_1', \ldots) = M f(a_0, a_1, \ldots),$$

where M is a function of l_1, m_1, l_2, m_2 only.

Now the equations (1) are easily proved to be the finite equations of a group of order four; but they are of little use in the invariant theory in comparison with their four infinitesimal operators

$$a_0 \frac{\partial}{\partial a_1} + 2 a_1 \frac{\partial}{\partial a_2} + 3 a_2 \frac{\partial}{\partial a_3} + \ldots,$$

$$a_p \frac{\partial}{\partial a_{p-1}} + 2 a_{p-1} \frac{\partial}{\partial a_{p-2}} + 3 a_{p-2} \frac{\partial}{\partial a_{p-3}} + \ldots,$$

$$a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + \ldots,$$

$$a_1 \frac{\partial}{\partial a_1} + 2 a_2 \frac{\partial}{\partial a_2} + 3 a_3 \frac{\partial}{\partial a_3} + \ldots.$$

A like result holds for most of the applications of continuous groups; thus, one of the questions to which the theory is applied is the investigation of those linear partial differential equations, which are unaltered by the transformations of a known group; we know that every equation, which admits all the infinitesimal transformations, will admit all the finite transformations of the group, for the latter can be thrown into canonical form; and it is much simpler to find the forms of differential equations admitting known infinitesimal transformations than the form of those admitting known finite transformations.

CHAPTER IV

THE CONDITIONS THAT A GIVEN SET OF LINEAR OPERATORS MAY GENERATE A GROUP

§ 47. We have proved in the last chapter that a group of order r has exactly r independent linear operators, in terms of which all other linear operators of the group can be expressed; and when these operators are known the group is also known in canonical form.

If $X_1, \ldots, X_r$ are *any* r independent operators of the group, we can express all other operators of the group in terms of these; there is therefore no unique system of operators; thus, in the group of rotations about the origin,

$$X \equiv y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y \equiv z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z \equiv x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

will be three independent operators; but so also would be

$$a_1X + b_1Y + c_1Z, \quad a_2X + b_2Y + c_2Z, \quad a_3X + b_3Y + c_3Z,$$

provided that the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

did not vanish.

We shall, however, suppose that we have fixed on some one set of independent operators, in terms of which the others are to be expressed.

The proposition which, with its converse, will form the subject of the present chapter may now be stated.

If $X_1, \ldots, X_r$ is a set of independent operators of the group, the alternant of any two of these is dependent on the set; that is

$$X_iX_j - X_jX_i \equiv (X_i, X_j) \equiv \sum^{k=r} c_{ijk}X_k, \qquad \binom{i = 1, \ldots, r}{j = 1, \ldots, r}$$

where the symbols $c_{ijk}, \ldots$ denote a set of constants, called the *structure constants* of the group; these constants are fixed,

when once the set $X_1, \ldots, X_r$ is fixed, but they vary with our choice of the set.

The converse of this theorem is, if $X_1, \ldots, X_r$ are any r independent linear operators such that

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k,$$

then $X_1, \ldots, X_r$ will be the operators of a group, which will be finite and continuous, and will contain the identical transformation; the canonical form of the group will be

$$x_i' = e^{e_1 X_1 + \ldots + e_r X_r} x_i, \qquad (i = 1, \ldots, n).$$

We have proved that in operating on any function of $x_1', \ldots, x_n', a_1, \ldots, a_r$, where we regard $x_1, \ldots, x_n$ as fixed, and $x_1', \ldots x_n'$ as varying, through being implicitly functions of $x_1, \ldots, x_n, a_1, \ldots, a_r$, we have (§ 40)

$$\frac{d}{da_k} = {}_aX_k' + \frac{\partial}{\partial a_k}.$$

Since then
$$\frac{d^2}{da_k da_h} = \frac{d^2}{da_h da_k},$$
we have

$$\left({}_aX_k' + \frac{\partial}{\partial a_k}\right)\left({}_aX_h' + \frac{\partial}{\partial a_h}\right) \equiv \left({}_aX_h' + \frac{\partial}{\partial a_h}\right)\left({}_aX_k' + \frac{\partial}{\partial a_k}\right);$$

expanding this we get

$$\begin{aligned}{}_aX_k'\, {}_aX_h' + {}_aX_k' \frac{\partial}{\partial a_h} + \frac{\partial}{\partial a_k}\, {}_aX_h' \\ \equiv {}_aX_h'\, {}_aX_k' + {}_aX_h' \frac{\partial}{\partial a_k} + \frac{\partial}{\partial a_h}\, {}_aX_k'.\end{aligned}$$

This identity is true for all values of $a_1, \ldots, a_r, x_1', \ldots, x_n'$; we may therefore replace x_i' by x_i, and in the notation of alternants we have

$$(1) \quad ({}_aX_k, {}_aX_h) + \left(\frac{\partial}{\partial a_k}, {}_aX_h\right) + \left({}_aX_k, \frac{\partial}{\partial a_h}\right) \equiv 0.$$

From the set of identities obtained in § 41, viz.

$${}_aX_k \equiv \lambda_{k1} X_1 + \ldots + \lambda_{kr} X_r$$

in which $\lambda_{ki}, \ldots$ only involve $a_1, \ldots, a_r$, we have

$$\left(\frac{\partial}{\partial a_k}, {}_aX_h\right) \equiv \frac{\partial \lambda_{h1}}{\partial a_k} X_1 + \ldots + \frac{\partial \lambda_{hr}}{\partial a_k} X_r,$$

$$\left({}_aX_k, \frac{\partial}{\partial a_h}\right) \equiv - \frac{\partial \lambda_{k1}}{\partial a_h} X_1 - \ldots - \frac{\partial \lambda_{kr}}{\partial a_h} X_r,$$

and therefore conclude from (1) that

$$({}_aX_k,\ {}_aX_h) \equiv \lambda_{kh1} X_1 + \dots + \lambda_{khr} X_r,$$

where the functions $\lambda_{khi}, \dots$ only involve $a_1, \dots, a_r$.

This identity holds for all values of the parameters $a_1, \dots, a_r$; we therefore take $a_1, \dots, a_r$ to be the parameters of the identical transformation, and the functions $\lambda_{khi}, \dots$ now become absolute constants and give the identities

$$(2) \qquad (X_i,\ X_j) = \sum^{k=r} c_{ijk} X_k.$$

This is called the second fundamental theorem in group theory.

Example. The equations (1) of § 46 are those of a group of order four, with the independent operators X_1, X_2, X_3, X_4, where

$$X_1 \equiv a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \dots + pa_{p-1} \frac{\partial}{\partial a_p},$$

$$X_2 \equiv pa_1 \frac{\partial}{\partial a_0} + (p-1)\, a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}},$$

$$X_3 \equiv a_0 \frac{\partial}{\partial a_0} + a_1 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_p},$$

$$X_4 \equiv a_1 \frac{\partial}{\partial a_1} + 2a_2 \frac{\partial}{\partial a_2} + \dots + pa_p \frac{\partial}{\partial a_p},$$

and we may verify that

$$(X_1, X_2) \equiv pX_3 - 2X_4, \quad (X_1, X_3) \equiv 0, \quad (X_1, X_4) \equiv X_1,$$
$$(X_2, X_3) \equiv -X_2, \quad (X_2, X_4) \equiv -X_2, \quad (X_3, X_4) \equiv 0.$$

If we take as the four independent operators of the group, Y_1, Y_2, Y_3, Y_4, where

$$Y_1 \equiv X_1, \quad Y_2 \equiv X_2, \quad Y_3 \equiv pX_3 - 2X_4, \quad Y_4 \equiv X_3,$$

we see that the group has the structure

$$(Y_1, Y_2) \equiv Y_3, \quad (Y_2, Y_3) \equiv 2Y_2, \quad (Y_3, Y_1) \equiv 2Y_1,$$
$$(Y_1, Y_4) \equiv 0, \quad (Y_2, Y_4) \equiv 0, \quad (Y_3, Y_4) \equiv 0.$$

§ 48. We now know that unless a system of linear operators is such that the alternant of any two of them is *dependent* on the set, they cannot generate a finite continuous group; but more important, and, at the same time, more difficult to prove, is the converse theorem, viz. that any operators which satisfy these conditions will generate a group.

Before proceeding to prove this we shall consider some formal laws according to which the symbols of linear operators are combined.

Let y and x denote two linear operators, and let y_1 denote $yx-xy$, y_2 denote y_1x-xy_1, y_3 denote y_2x-xy_2, and so on.

The identity

$$x^n y = yx^n - ny_1x^{n-1} + \frac{n(n-1)}{2!}y_2x^{n-2} - \ldots$$

may easily be proved by induction; for it is obviously true when $n = 1$; assume that it holds for all values up to n, then

$$x^{n+1}y = xyx^n - nxy_1x^{n-1} + \frac{n(n-1)}{2!}xy_2x^{n-2} - \ldots,$$

and as $xy_{r-1} = y_{r-1}x - y_r$, we have

$$\begin{aligned} x^{n+1}y &= yx^{n+1} - ny_1x^n + \frac{n(n-1)}{2!}y_2x^{n-1} - \ldots, \\ &\qquad - y_1x^n + ny_2x^{n-1} - \ldots, \\ &= yx^{n+1} - (n+1)y_1x^n + \frac{(n+1)(n)}{2!}y_2x^{n-1} - \ldots, \end{aligned}$$

so that the identity holds universally.

If we denote by $[y, x^r]$ the expression

$$yx^r + xyx^{r-1} + x^2yx^{r-2} + \ldots + x^ry,$$

we next prove the identity

$$\left[y, \frac{x^r}{(r+1)!}\right] = \frac{yx^r}{1!\,r!} - \frac{y_1x^{r-1}}{2!\,(r-1)!} + \frac{y_2x^{r-2}}{3!\,(r-2)!} - \ldots + (-1)^r\frac{y_r}{(r+1)!}.$$

Assuming that this identity holds for all values of r up to $n-1$, then

$$\begin{aligned} \left[y, \frac{x^n}{n!}\right] &= \left[y, \frac{x^{n-1}}{n!}\right]x + \frac{x^ny}{n!} \\ &= \frac{yx^n}{1!\,(n-1)!} - \frac{y_1x^{n-1}}{2!\,(n-2)!} + \ldots + (-1)^{n-1}\frac{y_{n-1}x}{n!} + \frac{x^ny}{n!}. \end{aligned}$$

Now we have proved that

$$x^ny = yx^n - ny_1x^{n-1} + \frac{n(n-1)}{2!}y_2x^{n-2} - \ldots;$$

so that by addition of similar terms in the two series we get

$$\left[y, \frac{x^n}{n!}\right] = (n+1)\left(\frac{yx^n}{1!\,n!} - \frac{y_1x^{n-1}}{2!\,(n-1)!} + \ldots + \frac{(-1)^ny_n}{(n+1)!}\right);$$

and as the identity holds when $n=1$ we conclude that it holds universally.

We have of course similarly

$$\left[y_s, \frac{x^r}{(r+1)!}\right] = \frac{y_s x^r}{1!\,r!} - \frac{y_{s+1} x^{r-1}}{2!\,(r-1)!} + \dots.$$

Examples.

(1) Prove similarly by induction the formula

$$yx^n = x^n y + nx^{n-1} y_1 + \frac{n(n-1)}{2!} x^{n-2} y_2 + \dots,$$

(2) If $y \equiv a_0 \dfrac{\partial}{\partial a_0} + 2a_1 \dfrac{\partial}{\partial a_1} + \dots + pa_{p-1} \dfrac{\partial}{\partial a_p}$,

$$x \equiv pa_1 \frac{\partial}{\partial a_0} + (p-1)\,a_2 \frac{\partial}{\partial a_1} + \dots + a_p \frac{\partial}{\partial a_{p-1}},$$

prove that

$$y_1 = p\left(a_0 \frac{\partial}{\partial a_0} + \dots + a_p \frac{\partial}{\partial a_p}\right)$$
$$-2\left(a_1 \frac{\partial}{\partial a_1} + 2a_2 \frac{\partial}{\partial a_2} + \dots + pa_p \frac{\partial}{\partial a_p}\right);$$

$$y(y_1+2) = y_1 y; \qquad y_2 = -2x, \quad y_3 = 0, \quad y_4 = 0, \dots.$$

(3) Prove that y and x being as defined in example (2),

$$yx^r = x^r y + rx^{r-1}(y_1 - r + 1),$$
$$y_1 y^r = y^r (y_1 + 2r).$$

*(4) Apply induction to deduce from (3) the more general formula

$$\frac{y^s x^r}{s!\,r!} = \frac{x^r}{r!}\frac{y^s}{s!} + \frac{x^{r-1}}{(r-1)!}\frac{y^{s-1}}{(s-1)!}(y_1 - r + s)$$
$$+ \frac{x^{r-2}}{(r-2)!}\frac{y^{s-2}}{(s-2)!}\frac{(y_1-r+s)}{1}\frac{(y_1-r+s-1)}{2}$$
$$+ \frac{x^{r-3}}{(r-3)!}\frac{y^{s-3}}{(s-3)!}\frac{(y_1-r+s)}{1}\frac{(y_1-r+s-1)}{2}\frac{(y_1-r+s-2)}{3} + \dots.$$

(5) Prove that x and y being any linear operators,

$$yx^2 - 2xyx + x^2 y$$

is a linear operator.

* A generalization of the formula of Hilbert, see Elliott, *Algebra of Quantics*, p. 154, Ex. 5.

(6) Prove that

$$y_r = yx^r - rxyx^{r-1} + \frac{r(r-1)}{2!}x^2yx^{r-2} - \dots + (-1)^r x^r y.$$

§ 49. Let

$$\frac{t}{e^t-1} = 1 - a_1t + a_2t^2 - a_3t^3 + a_4t^4 - a_5t^5 + a_6t^6 - \dots;$$

then, if $B_1, B_3, \dots$ are Bernouilli's numbers,

$$a_{2n} = (-1)^{n-1}\frac{B_{2n-1}}{(2n)!}, \quad \text{and} \quad a_3 = a_5 = a_7 = \dots = 0.$$

We shall now prove the identity

$$\frac{yx^r}{r!} = \left[y, \frac{x^r}{(r+1)!}\right] + a_1\left[y_1, \frac{x^{r-1}}{r!}\right] + a_2\left[y_2, \frac{x^{r-2}}{(r-1)!}\right] + \dots + a_r y_r.$$

If we substitute for each expression in brackets the series to which we have proved it equal, we find that the coefficient of yx^r on the right is $\frac{1}{r!}$, and that the coefficient of $y_s x^{r-s}$ is

$$(-1)^s\left(\frac{1}{(s+1)!\,(r-s)!} - \frac{a_1}{s!\,(r-s)!} + \frac{a_2}{(s-1)!\,(r-s)!} - \dots\right).$$

By equating the coefficients of the powers of t in

$$t = (e^t-1)(1 - a_1t + a_2t^2 - a_3t^3 + a_4t^4 - \dots)$$

we see that the expression in brackets is zero, and therefore the identity required is proved.

Example. If

$$z = y + a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + \dots$$

and $\quad z_r = z_{r-1}x - xz_{r-1}, \qquad (r = 1, 2, 3, \dots),$

prove that

$$y = z - \frac{z_1}{2!} + \frac{z_2}{3!} - \frac{z_3}{4!} + \dots.$$

We now let

$$z \equiv y + a_1y_1 + a_2y_2 + \dots \text{ to infinity,}$$

then, from what we have proved, we have

$$y = y,$$
$$yx = \tfrac{1}{2}[y, x] + a_1y_1,$$
$$y\frac{x^2}{2!} = \left[y, \frac{x^2}{3!}\right] + a_1\left[y_1, \frac{x}{2!}\right] + a_2y_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$y\frac{x^r}{r!} = \left[y, \frac{x^r}{(r+1)!}\right] + a_1\left[y_1, \frac{x^{r-1}}{r!}\right] + \dots.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding these expressions we get

$$(1)\quad ye^x = z + \left[z,\ \frac{x}{2\,!}\right] + \left[z,\ \frac{x^2}{3\,!}\right] + \dots \text{ to infinity.}$$

Now if t is a constant so small that its square may be neglected,

$$(x+ty)^r = x^r + t\,[y,\ x^{r-1}];$$

and therefore from (1), if we neglect t^2,

$$\begin{aligned}(1+ty)\,e^x &= 1 + x + tz + \frac{1}{2\,!}(x+tz)^2 + \frac{1}{3\,!}(x+tz)^3 + \dots \\ &= e^{x+tz}.\end{aligned}$$

We can now say that, if t is a constant no longer small,

$$(1+ty)\,e^x = e^{x+tz} + t^2 R,$$

where R is *some* operator formed by combinations of the symbols x and y.

§ 50. We now suppose that

$$\begin{aligned}x &\equiv e_1 X_1 + \dots + e_r X_r, \\ y &\equiv \epsilon_1 X_1 + \dots + \epsilon_r X_r,\end{aligned}$$

where $e_1, \dots, e_r$ and $\epsilon_1, \dots, \epsilon_r$ are two sets of parameters, and $X_1, \dots, X_r$ linear operators such that

$$(X_i,\ X_j) \equiv \sum^{k=r} c_{ijk} X_k, \qquad \begin{pmatrix} i = 1, \dots, r \\ j = 1, \dots, r\end{pmatrix}.$$

From these conditions it follows that, if z is the linear operator deduced from x and y by the law

$$z = y + a_1 y_1 + a_2 y_2 + \dots,$$

then z is equal to

$$c_1 X_1 + \dots + c_r X_r,$$

where $c_1, \dots, c_r$ are a set of constants, which are functions of $e_1, \dots, e_r$, $\epsilon_1, \dots, \epsilon_r$, and of the absolute constants $c_{ijk}, \dots$.

From the definition of z we see that these constants $c_1, \dots, c_r$ are analytic functions of $e_1, \dots, e_r$, $\epsilon_1, \dots, \epsilon_r$; and therefore the coefficients of the differential operators in z will be finite, provided that $e_1, \dots, e_r$, $\epsilon_1, \dots, \epsilon_r$ do not exceed certain fixed limits. It now follows that, e^x and e^{x+tz} being two operators whose effects on the subject of their operations are not in general infinite, the effect of R on any such subject cannot be infinite.

If we now denote by x_1 the operator $x + \dfrac{z}{m}$, where m is some integer, then x_1 will be a linear operator *dependent* on

$X_1, \ldots, X_r$; and the result at which we have arrived may be thus expressed

$$(1) \qquad \left(1+\frac{y}{m}\right)e^x = e^{x_1} + \frac{1}{m^2}R.$$

Similarly we must have

$$(2) \qquad \left(1+\frac{y}{m}\right)e^{x_1} = e^{x_2} + \frac{1}{m^2}R_1,$$

where x_1 has replaced x in (1).

So we have

$$(3) \qquad \left(1+\frac{y}{m}\right)e^{x_2} = e^{x_3} + \frac{1}{m^2}R_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\left(1+\frac{y}{m}\right)e^{x_{m-1}} = e^{x_m} + \frac{1}{m^2}R_{m+1}.$$

Multiplying (1) by $\left(1+\frac{y}{m}\right)^{m-1}$, (2) by $\left(1+\frac{y}{m}\right)^{m-2}$, and so on, and then adding we obtain

$$\left(1+\frac{y}{m}\right)^m e^x = e^{x_m} + \frac{1}{m^2}\left(\left(1+\frac{y}{m}\right)^{m-1}R + \left(1+\frac{y}{m}\right)^{m-2}R_1 + \ldots\right).$$

Now let m become infinite; from what we have proved for R we see that

$$\frac{1}{m^2}\left(\left(1+\frac{y}{m}\right)^{m-1}R + \left(1+\frac{y}{m}\right)^{m-2}R_1 + \ldots\right)$$

is an operator whose effect on any subject on which it operates is zero when m is infinite; and because x_m is always a linear operator dependent on $X_1, \ldots, X_r$ whatever m may be, and because also the limit of $\left(1+\frac{y}{m}\right)^m$ is e^y we conclude that

$$e^y e^x = e^X,$$

where X is *some* linear operator *dependent* on $X_1, \ldots, X_r$.

§ 51. We can now easily prove that a set of operators which have the property

$$(1) \qquad (X_i, X_j) = \sum^{k=r} c_{ijk} X_k$$

will generate a group.

From the definition of a group in canonical form, we see that what we have to prove is, that if

$$X \equiv \lambda_1 X_1 + \ldots + \lambda_r X_r,$$
$$Y \equiv \mu_1 X_1 + \ldots + \mu_r X_r,$$

where $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_r$ are two sets of parameters, and if Y' denotes the operator obtained by replacing x_i in Y by x'_i where
$$x'_i = e^X x_i, \qquad (i = 1, \ldots, n),$$
then
$$e^{Y'} x'_i = e^{\nu_1 X_1 + \ldots + \nu_r X_r} x_i,$$
where $\nu_1, \ldots, \nu_r$ are a set of parameters, which are functions of $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r$, and of the structure constants $c_{ijk}, \ldots$.

Now $e^{Y'} x'_i$ is a function of $x'_1, \ldots, x'_n$, and therefore by § 44, (4)
$$e^{Y'} x'_i = e^X . e^Y x_i;$$
and as we have proved that
$$e^X e^Y \equiv e^{\nu_1 X_1 + \ldots + \nu_r X_r},$$
we now conclude that the conditions (1) are sufficient as well as necessary in order that $X_1, \ldots, X_r$ may generate a group.

§ 52. To find $\nu_1, \ldots, \nu_r$ in terms of $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_r$ would be to find for the group in canonical form the functions
$$\phi_k(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r), \qquad (k = 1, \ldots, r),$$
which define the parameter groups.

Without attempting to perform the calculations necessary to find these functions, we can see the terms of the first degree in the expansions of $\nu_1, \ldots, \nu_r$ respectively, in powers of $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r$; for, neglecting all products of these parameters, we have
$$\begin{aligned} &e^{\lambda_1 X_1 + \ldots + \lambda_r X_r} e^{\mu_1 X_1 + \ldots + \mu_r X_r} \\ &\qquad = (1 + \lambda_1 X_1 + \ldots + \lambda_r X_r)(1 + \mu_1 X_1 + \ldots + \mu_r X_r), \\ &\qquad = 1 + (\lambda_1 + \mu_1) X_1 + \ldots + (\lambda_r + \mu_r) X_r, \end{aligned}$$
and therefore
$$\nu_k = \lambda_k + \mu_k + \ldots,$$
where the terms not written down are of higher degree than those which are written down.

It follows that any operation of the group
$$x'_i = e^{e_1 X_1 + \ldots + e_r X_r} x_i$$
can in general be written in the form
$$x'_i = e^{t_1 X_1} e^{t_2 X_2} \ldots e^{t_r X_r} x_i.$$

To prove this we recollect that the necessary and sufficient conditions, that r functions of r variables should be capable of assuming r assigned values, are that the functions should be unconnected. Now we have proved that $e^{t_1 X_1} e^{t_2 X_2} \ldots e^{t_r X_r}$ is equal to $e^{\nu_1 X_1 + \nu_2 X_2 + \ldots + \nu_r X_r}$,
where
$$\nu_k = t_k + \ldots, \qquad (k = 1, \ldots, r);$$

and as $t_1, \ldots, t_r$ are unconnected so must $v_1, \ldots, v_r$ be unconnected: by a suitable choice of the parameters $t_1, \ldots$ they can therefore be made to assume the respective values $e_1, \ldots, e_r$.

§ 53. *Example.* Prove that the operators

$$X \equiv y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y \equiv z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z \equiv x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

generate a group.

We have

$$(Y, Z) = -X, \quad (Z, X) = -Y, \quad (X, Y) = -Z,$$

and therefore by the converse of the second fundamental theorem these operators generate a group.

If now we require the equations of this particular group in finite form, we may proceed as follows.

The most general operation of the group is

$$e^{t_3 Z} e^{t_2 Y} e^{t_1 X}.$$

Let $\qquad x' = e^{t_1 X} x, \quad y' = e^{t_1 X} y, \quad z' = e^{t_1 X} z,$

so that

$$y' = \left(1 + t_1\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \frac{t_1^2}{2!}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)^2 + \ldots\right) y$$

$$= y - t_1 z - \frac{t_1^2}{2!} y + \frac{t_1^3}{3!} z + \frac{t_1^4}{4!} y - \ldots$$

$$= y \cos t_1 - z \sin t_1.$$

Similarly we see that

$$z' = y \sin t_1 + z \cos t_1$$

and $x' = x$.

We now have

$$e^{t_2 Y} e^{t_1 X} x = e^{t_2 Y'} x' = x' \cos t_2 + z' \sin t_2 = x'',$$

$$e^{t_2 Y} e^{t_1 X} y = e^{t_2 Y'} y' = y' \qquad\qquad = y'',$$

$$e^{t_2 Y} e^{t_1 X} z = e^{t_2 Z'} z' = z' \cos t_2 - x' \sin t_2 = z''.$$

And finally we get

$$x''' = x'' \cos t_3 - y'' \sin t_3,$$

$$y''' = x'' \sin t_3 + y'' \cos t_3.$$

From which equations we could express x''', y''', z''' in terms of x, y, z, and the parameters t_1, t_2, t_3.

§ 54. If m of the operators of a given group $X_1, \ldots, X_r$ are such that the alternant of any two of them is dependent on the m operators, then these m operators will themselves form a group, which will of course be a sub-group of $X_1, \ldots, X_r$.

Example. Find the projective transformations which do not alter the equation

$$x^2+y^2+z^2 = 1.$$

The most general operator of the projective group is

$$\left(a_0+a_1x+a_2y+a_3z+x(e_1x+e_2y+e_3z)\right)\frac{\partial}{\partial x}$$
$$+\left(b_0+b_1x+b_2y+b_3z+y(e_1x+e_2y+e_3z)\right)\frac{\partial}{\partial y}$$
$$+\left(c_0+c_1x+c_2y+c_3z+z(e_1x+e_2y+e_3z)\right)\frac{\partial}{\partial z};$$

we must therefore have

$$x(a_0+a_1x+a_2y+a_3z)+y(b_0+b_1x+b_2y+b_3z)$$
$$+z(c_0+c_1x+c_1y+c_3z)+(e_1x+e_2y+e_3z)(x^2+y^2+z^2) = 0$$

for all values of the variables such that

$$x^2+y^2+z^2 = 1.$$

This gives $\qquad a_1 = b_2 = c_3 = 0,$

$$a_2+b_1 = a_3+c_1 = b_3+c_2 = a_0+e_1 = b_0+e_2 = c_0+e_3 = 0,$$

so that there are six operators admitted by the given equation, viz.

$$X_1 = (x^2-1)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+xz\frac{\partial}{\partial z}, \quad Y_1 = y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},$$
$$X_2 = yx\frac{\partial}{\partial x}+(y^2-1)\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z}, \quad Y_2 = z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z},$$
$$X_3 = zx\frac{\partial}{\partial x}+zy\frac{\partial}{\partial y}+(z^2-1)\frac{\partial}{\partial z}, \quad Y_3 = x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}.$$

We find that

$$(X_2, X_3) = Y_1, \quad (X_3, X_1) = Y_2, \quad (X_1, X_2) = Y_3,$$
$$(Y_2, Y_3) = -Y_1, \quad (X_1, Y_1) = 0,$$
$$(Y_3, Y_1) = -Y_2, \quad (Y_1, Y_2) = -Y_3, \quad (X_1, Y_2) = -X_3,$$
$$(X_1, Y_3) = X_2, \quad (X_2, Y_2) = 0,$$
$$(X_2, Y_1) = X_3, \quad (X_2, Y_3) = -X_1, \quad (X_3, Y_1) = -X_2,$$
$$(X_3, Y_2) = X_1, \quad (X_3, Y_3) = 0;$$

these six operators will therefore generate a group, and of this group Y_1, Y_2, Y_3 will form a sub-group.

We could of course have foreseen that such operators must generate a group, from the general principle that if T_1 and T_2 are any two operators admitted by an equation, then $T_1 T_2$ is also admitted; and therefore the alternant $T_1 T_2 - T_2 T_1$, which is a linear operator, is also admitted; and must therefore be *connected* with the operators which belong to the group admitted by the equation.

Also in this example the group must be a finite one; for, if it is a group at all, it is a sub-group of the general projective group.

§ 55. If $X_1, \ldots, X_r$ are the operators of a simply transitive group, and $Y_1, \ldots, Y_s$ the operators of a second such group, and if the alternant of (X_i, Y_j) is zero for all values of i and j, then it is clear, from the canonical forms of the groups, that any operation of the one group is permutable with any operation of the other group; such groups are said to be *reciprocal.*

In the group we have just considered, taking as our set of six independent operators

$$Z_1 \equiv X_1 + iY_1, \quad Z_2 \equiv X_2 + iY_2, \quad Z_3 \equiv X_3 + iY_3,$$
$$W_1 \equiv X_1 - iY_1, \quad W_2 \equiv X_2 - iY_2, \quad W_3 \equiv X_3 - iY_3,$$

where i is a square root of negative unity, the group has, with respect to these operators, the structure

$$(Z_2, Z_3) = -2iZ_1, \quad (Z_3, Z_1) = -2iZ_2, \quad (Z_1, Z_2) = -2iZ_3,$$
$$(W_2, W_3) = 2iW_1, \quad (W_3, W_1) = 2iW_2, \quad (W_1, W_2) = 2iW_3,$$
$$(Z_i, W_j) = 0, \qquad \begin{pmatrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{pmatrix}.$$

It is easily proved that each of the sub-groups Z_1, Z_2, Z_3 and W_1, W_2, W_3 is simply transitive; they are therefore reciprocal sub-groups.

§ 56. *Examples.*

(1) If u, v, w are three quadratic functions of x, prove that

$$u\frac{\partial}{\partial x}, \quad v\frac{\partial}{\partial x}, \quad w\frac{\partial}{\partial x}$$

generate a group.

(2) Prove that $\quad \dfrac{\partial}{\partial x}$ and $x^3\dfrac{\partial}{\partial x}$

cannot be operators of a finite continuous group.

(3) Find the relations between the constants a, b, c, d in order that

$$(ax+by)\frac{\partial}{\partial x}+(cx+dy)\frac{\partial}{\partial y} \text{ and } x\frac{\partial}{\partial y}$$

may be operators of a group of order three.

(4) Prove that

$$y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y} \text{ and } (x^2-y^2-z^2)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}+2xz\frac{\partial}{\partial z}$$

are the operators of a group of order two; find the finite equations of the group, and hence verify that the group is an Abelian one.

(5) Prove that

$$y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y} \text{ and } (y^2-z^2-x^2)\frac{\partial}{\partial y}+2yx\frac{\partial}{\partial x}+2yz\frac{\partial}{\partial z}$$

are two operators of a group; and find the other operators of the group of lowest order containing these two.

§ 57. *Example.* Prove that a finite group containing

$$x\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial x},\quad y\frac{\partial}{\partial y},$$

cannot contain an operator of the form $u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$ where u and v are homogeneous integral functions of x and y, of degree higher than unity.

The principle which enables us to prove this theorem is that a group which contains two operators must contain their alternant. The alternant of two operators which are both homogeneous is then itself a homogeneous operator of the group; and if the degrees of the two operators are r and s the degree of the alternant is $(r+s-1)$. If then the group is to be finite, there must be a limit to the degree in which x and y can be involved in an operator; we may therefore suppose that there is no operator of degree higher than that of the operator

$$u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}.$$

Now suppose that $u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$ is of degree r, and can exist

in a group which contains

$$x\frac{\partial}{\partial x},\ x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ y\frac{\partial}{\partial y}.$$

As we cannot have u and v both identically zero, we may suppose that u is not identically zero.

Form the alternant of $u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$ with $x\frac{\partial}{\partial y}$, and we have an operator $u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}$ also of degree r; in u_1, however, y is of lower degree than it is in u.

By forming the alternant of $u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}$ with $x\frac{\partial}{\partial y}$, and proceeding similarly with the resultant operator, we see that the group must contain the operator

$$x^r\frac{\partial}{\partial x}+v\frac{\partial}{\partial y},$$

when v is some homogeneous function of x and y of degree r.

Denote this operator by Y, and $x\frac{\partial}{\partial x}$ by X, and let

$$Y_1 = YX-XY,\quad Y_2 = Y_1X-XY_1, \ldots$$

then $\quad Y_{r+1}\equiv(r-1)^{r+1}x^r\frac{\partial}{\partial x},\quad$ since $\quad x^{r+1}\frac{\partial v}{\partial x^{r+1}}\equiv 0.$

Now $r>1$: so that the group, if it exists, must contain the operator $x^r\frac{\partial}{\partial x}$.

Forming the alternant of $x^r\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial x}$, we see that the group will contain $yx^{r-1}\frac{\partial}{\partial x}$, and therefore

$$\left(x^r\frac{\partial}{\partial x},\ yx^{r-1}\frac{\partial}{\partial x}\right),$$

that is, $yx^{2r-2}\frac{\partial}{\partial x}$.

But, since $r>1$, this operator is of degree higher than r, and therefore we may conclude that the proposed group cannot exist.

§ 58. We proved in § 44 that $A_1, \ldots, A_r$, the operators of the first parameter group, were unconnected; and that $X_1, \ldots, X_r$ being the operators of the group of which $A_1, \ldots, A_r$ is the parameter group

$$X_1' + A_1, \ldots, X_r' + A_r$$

each annihilated any function of $x_1, \ldots, x_n$, when expressed in terms of $x_1', \ldots, x_n'$ and $a_1, \ldots, a_r$.

It follows that the alternant

$$(X_i' + A_i,\ X_j' + A_j)$$

annihilates such a function; and therefore so also does

$$(X_i' + A_i,\ X_j' + A_j) - \sum^{k=r} c_{ijk}\,(X_k' + A_k).$$

Expanding the alternant and noting that

$$(X_i',\ X_j') - \sum^{k=r} c_{ijk}\, X_k'$$

vanishes identically, we conclude that

$$(A_i,\ A_j) - \sum^{k=r} c_{ijk}\, A_k$$

annihilates any function of $x_1, \ldots, x_n$, when expressed in terms of $x_1', \ldots, x_n'$, $a_1, \ldots, a_r$.

Now this operator does not contain $x_1', \ldots, x_n'$, and therefore, from what we proved in § 42, it cannot annihilate the functions which express $x_1, \ldots, x_n$, respectively, in terms of $x_1', \ldots, x_n'$, $a_1, \ldots, a_r$, unless it vanishes identically; we must therefore conclude that

$$(A_i,\ A_j) \equiv \sum^{k=r} c_{ijk}\, A_k;$$

that is, *the first parameter group has the same structure constants as the group* $X_1, \ldots, X_r$.

§ 59. The theorem of § 41, known as the first fundamental theorem, tells us that if

$$(1) \quad x_i' = f_i(x_1, \ldots, x_n,\ a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

are the equations of a group, and

$${}_aX_1, \ldots, {}_aX_r$$

the operators derived from (1), by the method explained in § 40, then

(2) $\quad {}_aX_k = \lambda_{k1} X_1 + \ldots + \lambda_{kr} X_r, \qquad (k = 1, \ldots, r),$

where $\lambda_{kj}, \ldots$ are functions of $a_1, \ldots, a_r$, and

$$X_1, \ldots, X_r$$

are the operators obtained from

$$ {}_aX_1, \ldots, {}_aX_r$$

by substituting therein, for $a_1, \ldots, a_r$, the parameters of the identical transformation.

The converse of this theorem can now be proved.

Let (1) denote a system of equations known to involve the identical transformation; we can form the operators

$$ {}_aX_1, \ldots, {}_aX_r \text{ and } X_1, \ldots, X_r$$

from the equations (1) without presupposing any group property of those equations; the converse theorem then is, 'if the equations (2) are satisfied, then the equations (1) will define a finite continuous group.'

On referring back to § 44, it will be seen that the two facts, firstly that (1) involved the identical transformation, and secondly that its operators were connected by the equations (2), involved as a consequence that

$$x'_i = e^X x_i.$$

If therefore we can prove that the alternants obtained from $X_1, \ldots, X_r$ are dependent on $X_1, \ldots, X_r$, then the converse of the second fundamental theorem will show us that the equations (1) are the equations of a group.

Now the equations of § 40, viz.

(3) $$\frac{d}{da_k} = {}_aX'_k + \frac{\partial}{\partial a_k}, \qquad (k = 1, \ldots, r),$$

are independent of any group property in the equations (1); and (3) and (2) were the only equations used in § 47 to deduce (2) of that article. We conclude therefore that the facts, that

$$x'_i = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

involves the identical transformation, and that its operators are connected by the equations (2), are sufficient to ensure that the equations (1) are the equations of a group.

This is converse of the first fundamental theorem.

CHAPTER V

THE STRUCTURE CONSTANTS OF A GROUP

§ 60. If X_1, X_2, X_3 are any three linear operators whatever we have from the definition of an alternant

(1) $$(X_1, X_2)+(X_2, X_1) = 0.$$

Also from the same definition

$$\begin{aligned}(X_1, (X_2, X_3)) &= X_1(X_2, X_3)-(X_2, X_3)X_1 \\ &= X_1X_2X_3-X_1X_3X_2-X_2X_3X_1+X_3X_2X_1\end{aligned}$$

and therefore

(2) $$(X_1, (X_2, X_3))+(X_2, (X_3, X_1))+(X_3, (X_1, X_2)) = 0.$$

This equation will be referred to as Jacobi's identity.

If $X_1, \ldots, X_r$ are r independent operators the second fundamental theorem has shown us that

(3) $$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k,$$

if, and only if, these operators generate a group.

From (1) we then have

$$\sum^{k=r} (c_{ijk}+c_{jik}) X_k = 0;$$

and therefore, since the operators are independent,

$$c_{ijk}+c_{jik} = 0.$$

Again by (3) $(X_j, (X_i, X_k))$ is equal to

$$(X_j, \sum^{h=r} c_{ikh} X_h) = \sum^{h=r} c_{ikh}(X_j, X_h) = \sum^{h=m=r} c_{ikh}c_{jhm}X_m,$$

so that, applying Jacobi's identity, we have

$$\sum^{h=m=r}(c_{ikh}c_{jhm}+c_{kjh}c_{ihm}+c_{jih}c_{khm})X_m = 0.$$

Since the operators are independent we must therefore have

$$\sum^{h=r} (c_{ikh} c_{jhm} + c_{kjh} c_{ihm} + c_{jih} c_{khm}) = 0.$$

The constants then which occur in the identities

$$(X_i, X_k) = \sum^{h=r} c_{ikh} X_h$$

are such that they satisfy the system of equations

$$(4) \quad \begin{cases} c_{ikj} + c_{kij} = 0, \\ \sum\limits^{h=r} (c_{ikh} c_{jhm} + c_{kjh} c_{ihm} + c_{jih} c_{khm}) = 0, \end{cases}$$

where i, k, j, m may have any integral values from 1 to r.

These constants are the structure constants of the group corresponding to the operators $X_1, \ldots, X_r$.

The third fundamental theorem in the theory of finite continuous groups is that the structure constants of any group must satisfy these conditions; and the converse proposition is that any set of constants, satisfying these conditions, will be structure constants of some finite continuous group.

A set of constants satisfying the conditions (4) is called a set of structure constants of order r; what we are now about to show is, how, when we are given any such set of structure constants, r *unconnected* operators $X_1, \ldots, X_r$, in r variables, can be found such that

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k;$$

that is, we shall find r operators generating a simply transitive group, with the given constants as its structure constants.

Groups of order r with the given set of structure constants may exist in a number of variables greater or less than r; and the method of obtaining types of such groups will be investigated in Chapter XI; in this chapter, however, as we are only concerned to prove the converse of the third fundamental theorem, it will be sufficient to prove the existence of a simply transitive group with the required structure.

§ 61. If $\quad x_i = \sum^{k=r} a_{ki} x'_k, \qquad (i = 1, \ldots, r)$

is any linear transformation scheme, whose determinant

$$\begin{vmatrix} a_{11}, & \cdot & \cdot & \cdot & a_{1r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r1}, & \cdot & \cdot & \cdot & a_{rr} \end{vmatrix}$$

does not vanish, and $$x'_i = \sum^{k=r} A_{ki}\, x_k$$

is the inverse scheme, then, $c_{ikh}, \ldots$ being any other set of r^3 variables, and $c'_{ikh}, \ldots$ another set connected with the first set by the equation system

$$(1) \qquad \sum^{h=r} a_{hs}\, c'_{ikh} = \sum^{p=q=r} a_{ip}\, a_{kq}\, c_{pqs},$$

we see that, since the above determinant does not vanish, (1) must give $c'_{ikh}, \ldots$ in terms of $c_{ikh}, \ldots$.

From the fact that in the notation of § 38

$$\sum^{p=r} A_{pi}\, a_{kp} = \epsilon_{ik},$$

we easily verify that

$$\sum^{h=r} A_{hs}\, c_{ikh} = \sum^{p=q=r} A_{ip}\, A_{kq}\, c'_{pqs};$$

and therefore $c_{ikh}, \ldots$ are given in terms of $c'_{ikh}, \ldots$.

It will now be proved that if one set $c_{ikh}, \ldots$ satisfy the system of equations (4) of § 60, so will the other $c'_{ikh}, \ldots$.

To prove this, multiply (1) by $a_{tm}\, c_{smj}$, and sum for all values of h, s, m, p, q, when we shall have

$$\sum^{h=s=m=r} a_{hs}\, a_{tm}\, c'_{ikh}\, c_{smj} = \sum^{m=p=s=q=r} a_{ip}\, a_{kq}\, a_{tm}\, c_{pqs}\, c_{smj}.$$

Since by (1) the left hand member of this equation may be written $$\sum^{m=h=r} a_{mj}\, c'_{ikh}\, c'_{htm}$$

we see that

$$\sum^{m=h=r} a_{mj}\,(c'_{ikh}\, c'_{htm} + c'_{kth}\, c'_{him} + c'_{tih}\, c'_{hkm})$$

is the sum of a number of terms which vanish by the conditions (4) of § 60.

We therefore conclude, since the determinant does not vanish, that

$$\sum^{h=r} (c'_{ikh}\, c'_{htm} + c'_{kth}\, c'_{him} + c'_{tih}\, c'_{hkm}) = 0$$

for all values of i, k, m, t.

To prove that $$c'_{ikt} + c'_{kit} = 0,$$

interchange i, k, in (1); we then get

$$\sum^{h=r} a_{hs} c'_{kih} = \sum^{p=q=r} a_{iq} a_{kp} c_{pqs}.$$

Adding this equation to (1), from the conditions (4) of § 60 we must have

$$c'_{ikt} + c'_{kit} = 0.$$

Suppose now that we have a group with the structure constants $c_{ikh}, \ldots$, the corresponding operators being $X_1, \ldots, X_r$.

If we take as a new set of operators $Y_1, \ldots, Y_r$ where

$$(2) \qquad Y_i = \sum^{k=r} a_{ik} X_k,$$

then it can be at once verified that $c'_{ikh}, \ldots$ are the structure constants of the group corresponding to $Y_1, \ldots, Y_r$. The conclusion we draw is that when we can find a group with the structure constants $c_{ikh}, \ldots$ this group has also the structure constants $c'_{ikh}, \ldots$ corresponding to another set of independent operators.

We often take advantage of the fact that the structure constants of a group vary, with the choice of what we may call the fundamental set of operators, in order to simplify the structure constants of the group. Thus in § 55 we simplified the structure of the group of projective transformations admitted by $x^2 + y^2 + z^2 = 1$.

If two groups are such that the structure constants of the first, corresponding to some one fundamental set of operators, are the same as the structure constants of the second, corresponding to some one fundamental set of its operators, then the two groups are said to be of the *same structure.*

It is, however, a matter of considerable labour when we are given two groups, with their respective fundamental sets of operators not given in such a form as to have the same structure constants, to determine whether or no the groups have the same structure with respect to some two sets of fundamental operators.

§ 62. Suppose that we are given a set of structure constants $c_{ikh}, \ldots$ such that all $(r-s+1)$-rowed determinants, but not all $(r-s)$-rowed determinants, vanish in the matrix

$$\left\| \begin{matrix} c_{j1k}, \cdot & \cdot & \cdot \\ c_{j2k}, \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{jrk}, \cdot & \cdot & \cdot \end{matrix} \right\|$$

(in any row all positive integral values of j and k are to be taken from 1 to r).

We now choose constants $a_{ij}, \ldots$ such that

$$a_{h1}c_{j1k} + \ldots + a_{hr}c_{jrk} = 0,$$
$$(j = 1, \ldots, r;\ k = 1, \ldots, r;\ h = 1, \ldots, s),$$

and complete the determination of these constants by taking a_{mk} arbitrarily if $m > s$; these arbitrary constants, however, must be subject to the limitation that the determinant of the r^2 constants

$$\begin{vmatrix} a_{11}, & . & . & . & a_{1r} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{r1}, & . & . & . & a_{rr} \end{vmatrix} \neq 0.$$

If a group of the required structure exists, and $X_1, \ldots, X_r$ are its operators, then

$$a_{h1}X_1 + \ldots + a_{hr}X_r, \qquad (h = 1, \ldots, s)$$

will be s independent operators of the group permutable with every other operator of the group; that is, s Abelian operators forming therefore an Abelian sub-group.

We now take the operators given by (2) of § 61, and thus we get a new set of structure constants $c'_{ikh}, \ldots$ with the following properties:

(α) $$c'_{ikh} = d_{ikh}$$

where i, k, h may have any values from $(s+1)$ to r, and d_{ikh} are a set of structure constants of the n^{th} order, n being written for $(r-s)$;

(β) the constant $\quad c'_{ikh} = 0,$

if either i or k is less than $s+1$, h having any value from 1 to r inclusive;

(γ) the constants $\quad c'_{ikm}, \ldots$

where i and k both exceed s, and m does not exceed s, are such that

$$c'_{ikm} + c'_{kim} = 0,$$
$$\sum_{h=s+1}^{h=r} (d_{ikh}c'_{hjm} + d_{kjh}c'_{him} + d_{jih}c'_{hkm}) = 0.$$

We may therefore say (with the slight change of notation which consists in writing

$$d_{ikh} = c_{r-i,\, r-k,\, r-h}, \text{ and } c'_{ikm} = d'_{r-i,\, r-k,\, r-m})$$

that the problem of finding a group with the required structure is now reduced to that of finding a group with the structure constants $d'_{ikh}, \ldots$ defined by the following properties:

(α) $$d'_{ikh} = c_{ikh},$$

if none of the suffixes i, k, h exceeds n, where the constants c_{ikh} are known structure constants of the n^{th} order, such that not all n-rowed determinants vanish in the matrix

$$\begin{Vmatrix} c_{j1k}, & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{jrk}, & \cdot & \cdot & \cdot \end{Vmatrix}, \quad \begin{pmatrix} j = 1, \ldots, n \\ k = 1, \ldots, n \end{pmatrix};$$

(β) the constant $$d'_{ikh} = 0,$$

if either i or k exceeds n, h having any value from 1 to r;

(γ) $$d'_{ikm} = d_{ikm},$$

$$\text{where (1)} \begin{cases} d_{ikm} + d_{kim} = 0, \\ \sum\limits^{h=n} (c_{ikh} d_{hjm} + c_{kjh} d_{him} + c_{jih} d_{hkm}) = 0, \end{cases}$$

if neither i nor k exceeds n, and m does exceed n.

The constants $d'_{ikh}, \ldots$ may be called *normal structure constants*, and the problem of finding a group with a given set of structure constants is now reduced to that of finding a group with a given set of normal structure constants.

If $Y_1, \ldots, Y_r$ are the operators of a group with normal structure constants, $Y_{n+1}, \ldots, Y_r$ are the Abelian operators of the group, if any such exist; and there is no Abelian operator in the group *independent* of $Y_{n+1}, \ldots, Y_r$.

Example.

$$\begin{array}{lllll} c_{213} = -ce_3, & c_{312} = be_2, & c_{311} = be_1, & c_{113} = 0, & c_{112} = 0, \\ c_{223} = 0, & c_{322} = -ae_2, & c_{321} = -ae_1, & c_{123} = ce_3, & c_{122} = ce_2, \\ c_{233} = ae_3, & c_{332} = 0, & c_{331} = 0, & c_{133} = -be_3, & c_{132} = -be_2, \end{array}$$

$$\begin{array}{llll} c_{211} = -ce_1, & c_{111} = 0, & c_{212} = -ce_2, & c_{313} = be_3, \\ c_{221} = 0, & c_{121} = ce_1, & c_{222} = 0, & c_{323} = -ae_3, \\ c_{231} = ae_1, & c_{131} = -be_1, & c_{232} = ae_2, & c_{333} = 0, \end{array}$$

are a set of structure constants, forming the matrix

$$\begin{Vmatrix} -ce_3, & be_2, & be_1, & 0, & 0, & -ce_1, & 0, & -ce_2, & be_3 \\ 0, & -ae_2, & -ae_1, & ce_3, & ce_2, & 0, & ce_1, & 0, & -ae_3 \\ ae_3, & 0, & 0, & -be_3, & -be_2, & ae_1, & -be_1, & ae_2, & 0 \end{Vmatrix}.$$

We see that every determinant of the third order vanishes; and that, unless $a = b = c$, or $e_1 = e_2 = e_3 = 0$, it cannot happen that every determinant of the second degree vanishes.

If then a group X_1, X_2, X_3 exists with these given constants as structure constants,

$$aX_1 + bX_2 + cX_3$$

will be permutable with every operator of the group, that is, will be an Abelian operator; and we take then

$$Y_1 \equiv aX_1 + bX_2 + cX_3, \quad Y_2 = X_2, \quad Y_3 = X_3$$

to be the operators of the group.

We have now a group of which the structure is

$$(Y_2, Y_3) = e_1 Y_1 + (ae_2 - be_1) Y_2 + (ae_3 - ce_1) Y_3$$
$$(Y_1, Y_3) = 0, \qquad (Y_1, Y_2) = 0.$$

If $ae_2 - be_1$, and $ae_3 - ce_1$ are both zero, we see that $Z_1 = eY_1$, $Z_2 = Y_2$, $Z_3 = Y_3$ will be three independent operators of the group with the structure

$$(Z_1, Z_2) = 0, \quad (Z_1, Z_3) = 0, \quad (Z_2, Z_3) = Z_1.$$

If $ae_2 - be_1$ and $ae_3 - ce_1$ are not both zero, suppose that $ae_2 - be_1$ is not zero, and take

$$Z_2 = e_1 Y_1 + (ae_2 - be_1) Y_2 + (ae_3 - ee_1) Y_3, \quad Z_3 = (ae_2 - be_1)^{-1} Y_3,$$

when we shall have

$$(Z_2, Z_3) = Z_2, \quad (Z_1, Z_2) = 0, \quad (Z_1, Z_3) = 0.$$

§ 63. We have proved in § 58 that the first parameter group has the same structure constants as the group which generates it, and that it is a simply transitive group. Now it may be at once verified that, if

$$X_i = \sum^{j=k=n} c_{jik}\, x_j \frac{\partial}{\partial x_k}, \qquad (i = 1, \ldots, n),$$

then the operators $X_1, \ldots, X_n$, if independent, will form a linear group with the structure constants $c_{jik}, \ldots$. The first parameter group of this linear group will be simply transitive and have these constants as its structure constants.

Now the operators $X_1, \ldots, X_n$ are independent, since by hypothesis not all n-rowed determinants vanish in the matrix

$$\left\| \begin{matrix} c_{j1k}, & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{jrk}, & \cdot & \cdot & \cdot \end{matrix} \right\|;$$

and we thus see that, given the structure constants, the group can be at once obtained if it does not contain any Abelian operators.

Example. Find a simply transitive group with the structure

$$c_{121} = 1, \quad c_{122} = 0, \quad c_{211} = -1, \quad c_{212} = 0, \quad c_{111} = 0,$$
$$c_{112} = 0, \quad c_{221} = 0, \quad c_{222} = 0.$$

Writing down the matrix we see that

$$X_1 \equiv x\frac{\partial}{\partial x}, \qquad X_2 \equiv y\frac{\partial}{\partial x}$$

is a group of the required structure, but it is not simply transitive.

The finite equations of this group in canonical form are (if we take $e_1 X_1 + e_2 X_2$ as the general operator of the group)

$$x' = e^{e_1 t} x + \frac{e_1}{e_2}(e^{e_1 t} - 1)\, y, \qquad y' = y.$$

If we change to a new set of parameters given by

$$a_1 = e^{e_1 t}, \qquad a_2 = \frac{e_1}{e_2}(e^{e_1 t} - 1)$$

the finite equations of the group are no longer given in canonical form, but yet they take the simple form

$$x' = a_1 x + a_2 y, \qquad y' = y.$$

The first parameter group is now

$$x' = a_1 x, \qquad y' = a_1 y + a_2,$$

since the equations which generate it are

$$c_1 = a_1 b_1, \qquad c_2 = b_1 a_2 + b_2.$$

The parameter group is therefore a group of the required type, since it is simply transitive, and it may be verified that it has the required structure, for its operators are

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial y}.$$

§ 64. We now proceed with the theory of the construction of a group when the assigned structure constants are such that the group, if it exists, must contain Abelian operators.

Let $X_1, \ldots, X_n$ be the simply transitive group, which we have shown how to construct with the structure constants $c_{ikh}, \ldots$.

Assuming for the moment that the simultaneous equation system

$$(1) \qquad X_i u_{km} - X_k u_{im} = d_{ikm} + \sum^{h=n} c_{ikh} u_{hm},$$

$$(i = 1, \ldots, n;\ k = 1, \ldots, n;\ m = n+1, \ldots, r)$$

can be solved, let $u_{1m}, \ldots, u_{nm}$ be any set of integrals. We can then at once verify that the r linear operators

$$X_i + u_{i,\,n+1} \frac{\partial}{\partial x_{n+1}} + \ldots + u_{ir} \frac{\partial}{\partial x_r}, \qquad (i = 1, \ldots, n),$$

$$\frac{\partial}{\partial x_{n+1}}, \ldots, \frac{\partial}{\partial x_r}$$

generate a simply transitive group of order r with the structure constants $d'_{ikh}, \ldots$.

Example. Find a group with the structure

$$(X_2, X_4) = 0, \quad (X_1, X_2) = -X_2 + X_3, \quad (X_1, X_3) = 0,$$
$$(X_2, X_3) = 0, \quad (X_3, X_4) = 0, \quad (X_1, X_4) = 0.$$

The constants of the proposed group are such that the group must have two Abelian operators; and the constants are in normal form, for X_3 and X_4 are clearly these Abelian operators.

Using the results of the last example, we take

$$X_1 \equiv x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \qquad X_2 = \frac{\partial}{\partial x_2},$$

and the operators of the required group will be Y_1, Y_2, Y_3, Y_4,

where $$Y_1 = X_1 + \xi_3 \frac{\partial}{\partial x_3} + \xi_4 \frac{\partial}{\partial x_4}, \quad Y_2 = X_2 + \eta_3 \frac{\partial}{\partial x_3} + \eta_4 \frac{\partial}{\partial x_4},$$

$$Y_3 = \frac{\partial}{\partial x_3}, \quad Y_4 = \frac{\partial}{\partial x_4}.$$

We see then, by the condition of the problem, that $\xi_3, \xi_4, \eta_3, \eta_4$ are functions not involving x_3 or x_4, and that

$$X_1 \eta_3 - X_2 \xi_3 = 1 - \eta_3, \qquad X_1 \eta_4 - X_2 \xi_4 = 0.$$

As we can take any integrals of these equations, we choose $\eta_4 = \xi_4 = \xi_3 = 0$; and we must then determine η_3 so that

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)\eta_3 = 1 - \eta_3.$$

We therefore take $\eta_3 = 1$, and we see that

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \qquad \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \qquad \frac{\partial}{\partial x_3}, \qquad \frac{\partial}{\partial x_4}$$

will be four independent operators forming a group of the required structure.

§ 65. We now proceed to show how the equation system (1) of § 64 may be solved.

Since $X_1, \ldots, X_n$ is a known set of unconnected operators, $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ can be expressed thus:—

$$\frac{\partial}{\partial x_i} = \lambda_{1i} X_1 + \ldots + \lambda_{ni} X_n, \qquad (i = 1, \ldots, n),$$

where $\lambda_{ik}, \ldots$ is a known set of functions of the variables $x_1, \ldots, x_n$.

From the fact that

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}$$

and that $X_1, \ldots, X_n$ form a group, we see that $\lambda_{ik}, \ldots$ are functions satisfying the equation system

$$(1) \qquad \frac{\partial \lambda_{jk}}{\partial x_i} - \frac{\partial \lambda_{ji}}{\partial x_k} = \sum^{\alpha=\beta=n} c_{\alpha\beta j} \lambda_{\alpha k} \lambda_{\beta i}.$$

It will now be verified that

$$(2) \quad \sum^{\alpha=\beta=n} d_{\alpha\beta m}\left(\frac{\partial}{\partial x_i} \lambda_{\alpha j} \lambda_{\beta k} + \frac{\partial}{\partial x_j} \lambda_{\alpha k} \lambda_{\beta i} + \frac{\partial}{\partial x_k} \lambda_{\alpha i} \lambda_{\beta j}\right) \equiv 0$$

for all values of i, j, k.

We have

$$\frac{\partial}{\partial x_i} \lambda_{\alpha j} \lambda_{\beta k} = \lambda_{\alpha j} \frac{\partial}{\partial x_i} \lambda_{\beta k} + \lambda_{\beta k} \frac{\partial}{\partial x_i} \lambda_{\alpha i},$$

$$\frac{\partial}{\partial x_j} \lambda_{\alpha k} \lambda_{\beta i} = \lambda_{\alpha k} \frac{\partial}{\partial x_j} \lambda_{\beta i} + \lambda_{\beta i} \frac{\partial}{\partial x_j} \lambda_{\alpha k},$$

$$\frac{\partial}{\partial x_k} \lambda_{\alpha i} \lambda_{\beta j} = \lambda_{\alpha i} \frac{\partial}{\partial x_k} \lambda_{\beta j} + \lambda_{\beta j} \frac{\partial}{\partial x_k} \lambda_{\alpha i}.$$

Since $$d_{\alpha\beta m}+d_{\beta\alpha m}=0,$$
we see that what we have to prove is that

$$\sum^{\alpha=\beta=n}\lambda_{\alpha j}d_{\alpha\beta m}\Big(\frac{\partial}{\partial x_i}\lambda_{\beta k}-\frac{\partial}{\partial x_k}\lambda_{\beta i}\Big)+\sum^{\alpha=\beta=n}\lambda_{\beta k}d_{\alpha\beta m}\Big(\frac{\partial}{\partial x_i}\lambda_{\alpha j}-\frac{\partial}{\partial x_j}\lambda_{\alpha i}\Big)$$
$$+\sum^{\alpha=\beta=n}\lambda_{\beta i}d_{\alpha\beta m}\Big(\frac{\partial}{\partial x_j}\lambda_{\alpha k}-\frac{\partial}{\partial x_k}\lambda_{\alpha j}\Big)\equiv 0.$$

Writing the second and third of these sums in the respectively equivalent forms,

$$\sum^{\gamma=\beta=n}\lambda_{\gamma k}d_{\gamma\beta m}\Big(\frac{\partial}{\partial x_j}\lambda_{\beta i}-\frac{\partial}{\partial x_i}\lambda_{\beta j}\Big),$$
$$\sum^{b=\beta=n}\lambda_{bi}d_{b\beta m}\Big(\frac{\partial}{\partial x_k}\lambda_{\beta j}-\frac{\partial}{\partial x_j}\lambda_{\beta k}\Big),$$

and substituting from (1), we see that the coefficient of $\lambda_{\alpha j}\lambda_{\gamma k}\lambda_{bi}$ in the identity is

$$-\sum^{\beta=n}(d_{\beta\alpha m}c_{\gamma b\beta}+d_{\beta a m}c_{b\gamma\beta}+d_{\beta bm}c_{\alpha\gamma\beta});$$

and this is zero by (1) of § 62, so that the identical relation (2) is now proved.

In order to prove that the simultaneous equation system (1) of § 64 can be satisfied, multiply the equation there given by $\lambda_{ip}\lambda_{kq}$, and sum for all values of i, k; then, if the new set of equations—there will be one for each pair of values of p, q—can be satisfied, so can the old.

To see this we notice that for the equation, with a given pair of values of i, k, the multiplier is $\lambda_{ip}\lambda_{kq}-\lambda_{kp}\lambda_{iq}$; and the determinant of these multipliers cannot vanish, for the determinant of λ_{pq} does not vanish (Forsyth, *Differential Equations*, § 212).

If we now take

$$v_{im}=\lambda_{1i}u_{1m}+\ldots+\lambda_{ni}u_{nm}, \qquad (i=1,\ldots,n),$$
the simultaneous equation system takes the simple form

$$\frac{\partial}{\partial x_p}\cdot v_{qm}-\frac{\partial}{\partial x_q}\cdot v_{pm}=\sum^{i=k=n}d_{ikm}\lambda_{ip}\lambda_{kq}=\sigma_{pqm},$$
where $\sigma_{ikm},\ldots$ are functions such that
$$\sigma_{ikm}+\sigma_{kim}=0,$$
since $$d_{ikm}+d_{kim}=0;$$

and from (2) we know that

$$\frac{\partial}{\partial x_i}\sigma_{jkm}+\frac{\partial}{\partial x_j}\sigma_{kim}+\frac{\partial}{\partial x_k}\sigma_{ijm}=0.$$

§ 66. To solve these equations consider the following lemma: if we have $\frac{1}{2}n(n-1)$ functions $\sigma_{ik}, \ldots$ of the variables $x_1, \ldots, x_n$ such that

$$\sigma_{ik}+\sigma_{ki}=0,$$

$$\frac{\partial}{\partial x_i}\sigma_{jk}+\frac{\partial}{\partial x_j}\sigma_{ki}+\frac{\partial}{\partial x_k}\sigma_{ij}=0,$$

$$(i=1, \ldots, n;\ j=1, \ldots, n;\ k=1, \ldots, n),$$

then n functions $u_1, \ldots, u_n$ can be found such that

$$\sigma_{ik}=\frac{\partial^2}{\partial x_i\,\partial x_k}(u_i-u_k).$$

To prove that this is true for the case $n=3$, let

$$\sigma_{12}=\frac{\partial^2}{\partial x_1\,\partial x_2}(u_1-u_2), \qquad \sigma_{13}=\frac{\partial^2}{\partial x_1\,\partial x_3}(u_1-u_3);$$

here we can take u_1 arbitrarily, and obtain u_2 and u_3 by integration.

Since $\sigma_{12}+\sigma_{21}=0$, and $\sigma_{13}+\sigma_{31}=0$,

$$\sigma_{21}=\frac{\partial^2}{\partial x_1\,\partial x_2}(u_2-u_1), \qquad \sigma_{31}=\frac{\partial^2}{\partial x_1\,\partial x_3}(u_3-u_1).$$

Now $$\frac{\partial}{\partial x_1}\sigma_{23}+\frac{\partial}{\partial x_2}\sigma_{31}+\frac{\partial}{\partial x_3}\sigma_{12}=0,$$

therefore $$\frac{\partial}{\partial x_1}\sigma_{23}+\frac{\partial^3}{\partial x_1\,\partial x_2\,\partial x_3}(u_3-u_2)=0,$$

and therefore

$$\sigma_{23}=\frac{\partial^2}{\partial x_2\,\partial x_3}(u_2-u_3)+f(x_2, x_3).$$

It is clear that we can write $f(x_2, x_3)$ in the equivalent form

$$f(x_2, x_3)=\frac{\partial^2}{\partial x_2\,\partial x_3}(w_2-w_3)$$

where w_2 and w_3 are functions of x_2, x_3 only; and if w_2 is taken to be some arbitrary function, then w_3 can be obtained by integration; therefore

$$\sigma_{23}=\frac{\partial^2}{\partial x_2\,\partial x_3}(u_2+w_2-u_3-w_3).$$

Since w_2 and w_3 do not involve x_1, we see that u_1, u_2+w_2, and u_3+w_3 are three functions in terms of which σ_{23}, σ_{31}, and σ_{12} can be expressed in the required form.

The extension to n variables is now easy. Assuming that the theorem has been proved for the case of $(n-1)$ variables,

let $$\sigma_{1k} = \frac{\partial^2}{\partial x_1 \partial x_k}(u_1-u_k), \qquad (k = 1, \ldots, n),$$

where as before u_1 is arbitrary.

From $$\frac{\partial}{\partial x_1}\sigma_{kh} + \frac{\partial}{\partial x_k}\sigma_{h1} + \frac{\partial}{\partial x_h}\sigma_{1k} = 0,$$

we get $$\frac{\partial}{\partial x_1}\sigma_{kh} = \frac{\partial_3}{\partial x_1 \partial x_k \partial x_h}(u_k-u_h),$$

and therefore $$\sigma_{kh} = \frac{\partial^2}{\partial x_k \partial x_h}(u_k-u_h)+\rho_{kh},$$

where ρ_{kh} is a function of $x_2, \ldots, x_n$ only.

We have $$\rho_{kh}+\rho_{hk} = 0,$$

$$\frac{\partial}{\partial x_i}\rho_{hk} + \frac{\partial}{\partial x_h}\rho_{ki} + \frac{\partial}{\partial x_k}\rho_{ih},$$

$$(i = 2, \ldots, n;\ h = 2, \ldots, n;\ k = 2, \ldots, n);$$

and therefore, since we now have only $(n-1)$ variables,

$$\rho_{kh} = \frac{\partial^2}{\partial x_h \partial x_k}(w_k-w_h),$$

where $w_2, \ldots, w_n$ do not involve x_1.

It follows as before that

$$u_1,\ u_2+w_2,\ \ldots,\ u_n+w_n$$

will be a set of functions in terms of which we can express $\sigma_{ik}, \ldots$ in the required manner.

If we now write, as we can,

$$\sigma_{pqm} = \frac{\partial^2}{\partial x_p \partial x_q}(V_{pm}-V_{qm}),$$

where the functions $V_{pm}, \ldots$ can be obtained by quadrature, the integrals of the equation system,

$$\frac{\partial}{\partial x_p}v_{qm} - \frac{\partial}{\partial x_q}v_{pm} = \sigma_{pqm},$$

will be $$v_{pm} = -\frac{\partial}{\partial x_p}V_{pm}.$$

§ 67. We have thus proved that, given any set of structure constants, we can in all cases find a simply transitive group of that structure.

Of the three fundamental theorems in the theory of finite continuous groups, the first asserts that in a group with r parameters there are exactly r operators which are independent; and this property, together with the existence of the identical transformation, is sufficient to ensure that the equations

$$x'_i = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i=1, \ldots, n)$$

will define a group.

The second fundamental theorem asserts that these operators $X_1, \ldots, X_r$ are such that

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k;$$

and that from any set of linear operators satisfying these identities a group may be generated. The theory of the canonical form of a group shows us that the group is entirely given, when we know the linear operators; and therefore, to find all possible groups, we have to find all possible sets of independent operators, such that the alternants of any set are *dependent* on the operators of that set.

The third fundamental theorem asserts that this set of structure constants satisfies the conditions

$$c_{ikh} + c_{kih} = 0,$$

$$\sum^{h=r} (c_{ikh} c_{jhm} + c_{kjh} c_{ihm} + c_{jih} c_{khm}) = 0;$$

and that, corresponding to every set of constants satisfying these conditions, a simply transitive group can be found whose operators satisfy the conditions

$$(X_i, X_k) = \sum^{h=r} c_{ikh} X_h.$$

Later on we shall see how all types of groups with a given set of constants as structure constants can be found, for so far the third fundamental theorem has merely shown us that one simply transitive group of the required structure may be found.

CHAPTER VI

COMPLETE SYSTEMS OF DIFFERENTIAL EQUATIONS

§ 68. If q linear operators $X_1, \ldots, X_q$ are such that no identity of the form

$$\phi_1(x_1, \ldots, x_n) X_1 + \ldots + \phi_q(x_1, \ldots, x_n) X_q \equiv 0$$

connects them, the operators are said to be *unconnected*.

Any operator which can be expressed in the form

$$\phi_1(x_1, \ldots, x_n) X_1 + \ldots + \phi_q(x_1, \ldots, x_n) X_q$$

is said to be connected with $X_1, \ldots, X_q$; and all operators so connected are said to belong to the system $X_1, \ldots, X_q$.

There cannot be more than n unconnected operators, though there may be an infinity of independent operators; unconnected operators are of course independent, but independent operators may be connected (§ 15).

If $\phi_1(x_1, \ldots, x_n)$ and $\phi_2(x_1, \ldots, x_n)$ are two functions of the variables $x_1, \ldots, x_n$, such that there is no functional relation between them of the form

$$\psi(\phi_1, \phi_2) \equiv 0,$$

they are generally said to be independent; it will be perhaps more convenient if we say they are *unconnected*, and reserve the word *independent* for functions not connected by a relation of the form

$$\lambda_1 \phi_1 + \lambda_2 \phi_2 \equiv 0,$$

where λ_1 and λ_2 are constants, and not both zero.

Similarly any number of functions $\phi_1, \ldots, \phi_s$ will be said to be *unconnected* if there is no identical relation between them of the form

$$\psi(\phi_1, \ldots, \phi_s) \equiv 0;$$

and they will be said to be *independent* if there is no relation between them of the form

$$\lambda_1 \phi_1 + \ldots + \lambda_s \phi_s \equiv 0,$$

where $\lambda_1, \ldots, \lambda_s$ are constants.

If we have q unconnected operators, such that the alternant of any pair is connected with the q operators; that is, if

$$(X_i, X_k) = \sum^{h=q} \phi_h X_h,$$

the operators are said to form a *complete system* of order q.

If we take any system of unconnected operators $X_1, \ldots, X_q$, and form their alternants $(X_i, X_k), \ldots$, then, unless each alternant is connected with $X_1, \ldots, X_q$, the system made up of $X_1, \ldots, X_q$ and their alternants $(X_i, X_k), \ldots$ will contain a greater number of unconnected operators than the original system $X_1, \ldots, X_q$.

Suppose it contains $(q+s)$ unconnected operators; we can add to this system as we added to the original system, and we shall thus obtain a new system containing still more unconnected operators; proceeding in this way we must at last arrive at a complete system, since there can never be more unconnected operators than there are variables.

If a function of $x_1, \ldots, x_n$ is unaltered by the infinitesimal transformation

$$x_i' = x_i + t\,\xi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

it is said to *admit* the infinitesimal transformation, or to be an invariant of that transformation.

If $f(x_1, \ldots, x_n)$ is a function admitting this transformation we must have

$$f(x_1, \ldots, x_n) = f(x_1', \ldots, x_n') = f(x_1, \ldots, x_n) + t \sum^{i=n} \xi_i \frac{\partial f}{\partial x_i};$$

it follows that the necessary and sufficient condition that the function may admit the infinitesimal transformation is that it should be annihilated by the linear operator

$$\xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_n \frac{\partial}{\partial x_n}.$$

The set of q infinitesimal transformations

$$x_i' = x_i + t\,\xi_{ki}(x_1, \ldots, x_n), \qquad \binom{k = 1, \ldots, q}{i = 1, \ldots, n}$$

are said to be *unconnected* if no identities of the form

$$\sum^{k=q} \phi_k \cdot \xi_{ki}(x_1, \ldots, x_n) \equiv 0, \qquad (i = 1, \ldots, n)$$

connect them, where $\phi_1, \ldots, \phi_q$ are functions of the variables $x_1, \ldots, x_n$.

§ 69. The problem of finding whether there is any function $f(x_1, \ldots, x_n)$ admitting a given set of q unconnected infinitesimal transformations, is the same as that of finding whether there is any function annihilated by each of the q given operators

$$X_1, \ldots, X_q.$$

Since, if f is annihilated by X_i and X_j, it is also annihilated by the alternant (X_i, X_j), this problem may be replaced by that of finding whether there is any function annihilated by the operators of a complete system.

If the complete system is of order n, i.e. if the number of unconnected operators is equal to the number of variables, then the only function which can be so annihilated is a mere constant.

If, however, the order is less than n, it will now be proved that there are $(n-q)$ functions which are so annihilated; in other words, *there are $(n-q)$ unconnected invariants of a complete system of order q.*

Let $Y_1, \ldots, Y_q$ be a new set of operators connected with $X_1, \ldots, X_q$ by the identities

$$Y_k \equiv \rho_{k1} X_1 + \ldots + \rho_{kq} X_q, \qquad (k = 1, \ldots, q),$$

where $\rho_{ik}, \ldots$ are any system of functions such that the determinant

$$\begin{vmatrix} \rho_{11}, & \cdot & \cdot & \cdot & \rho_{1q} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{q1}, & \cdot & \cdot & \cdot & \rho_{qq} \end{vmatrix}$$

is not identically zero.

The operators $Y_1, \ldots, Y_q$ also form a complete system of order q, and any invariant of one system is an invariant of the other.

In order to simplify the forms of $Y_1, \ldots, Y_q$ we now so choose $\rho_{ik}, \ldots$ as to have, in the notation of § 38,

$$\sum_{i=1}^{i=q} \rho_{ki}\, \xi_{i\,n-q+h} \equiv \epsilon_{kh}.$$

Since $X_1, \ldots, X_q$ are unconnected, these values of $\rho_{ik}, \ldots$ cannot make the above determinant vanish; we now have

$$Y_k \equiv \frac{\partial}{\partial x_{n-q+k}} + \sum^{i=n-q} \eta_{ki}(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \qquad (k = 1, \ldots, q).$$

The operators $Y_1, \ldots, Y_q$ are now said to be in *normal form*, and the problem before us is to find the unconnected invariants of a complete system given in normal form.

The operators in normal form are all permutable; for suppose that

$$(Y_i, Y_k) \equiv \mu_1 Y_1 + \ldots + \mu_q Y_q$$

where $\mu_1, \ldots, \mu_q$ are functions of $x_1, \ldots, x_n$.

From the forms of $Y_1, \ldots, Y_q$ we see that the coefficient of $\dfrac{\partial}{\partial x_{n-q+h}}$ in the alternant of (Y_i, Y_k) is zero; and, since on the right hand of the above identity this coefficient is μ_h, we conclude that $\mu_1, \ldots, \mu_q$ are each zero.

We now know that $Y_1, \ldots, Y_q$ generate an Abelian group, all of whose operators are unconnected. (It is not of course true that the operators $X_1, \ldots, X_q$ necessarily generate a group; such a conclusion could only be drawn if $X_1, \ldots, X_q$ were *dependent* on $Y_1, \ldots, Y_q$; here all we know is that they are *connected* with $Y_1, \ldots, Y_q$.)

The problem of finding the integrals of a complete system of linear partial differential equations is the same as that of finding the invariants of the corresponding operators; and this problem is now reduced to that of determining the invariants of a known Abelian group, all of whose operators are *unconnected*.

It will be noticed that in this reduction of the problem only the direct processes of algebra have so far been employed.

§ 70. We shall now show how the form of such an Abelian group may be simplified by the introduction of new variables.

Let $$X = \xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_n \frac{\partial}{\partial x_n}$$

be any operator, and let $f_1(x_1, \ldots, x_n), \ldots, f_{n-1}(x_1, \ldots, x_n)$ be any $(n-1)$ unconnected invariants of this operator, and $f_n(x_1, \ldots, x_n)$ any other function *unconnected* with $f_1, \ldots, f_{n-1}$.

Take as a new set of variables

$$y_1 = f_1, \ldots, y_n = f_n:$$

then the operator X, when expressed in terms of these new variables, must be of the form $\eta \dfrac{\partial}{\partial y_n}$, where η is some function of $y_1, \ldots, y_n$, which is known, when we know X and its invariants.

We can now find by quadratures a function $\phi(y_1, \ldots, y_n)$ such that

$$\eta \frac{\partial \phi}{\partial y_n} = 1.$$

This function ϕ, which we shall now denote by y'_n, must contain y_n, and must therefore be unconnected with $y_1, \ldots, y_{n-1}$; if then we take as variables $y_1, \ldots, y_{n-1}, y'_n$, the operator X will be of the form

$$\frac{\partial}{\partial y'_n}.$$

In order to bring Y_q into the form $\frac{\partial}{\partial y'_n}$, it is only necessary to be able to find the invariants of a linear operator in $(n-q+1)$ variables; for, since the coefficients of

$$\frac{\partial}{\partial x_{n-q+1}}, \ldots, \frac{\partial}{\partial x_{n-1}}$$

vanish in Y_q, the variables $x_{n-q+1}, \ldots, x_{n-1}$ can only enter that operator in the form of parameters.

(It is not to be supposed that in every operator of any Abelian group the coefficients of $\frac{\partial}{\partial x_{n-q+1}}, \ldots, \frac{\partial}{\partial x_{n-1}}$ must vanish; but in the particular Abelian group we are dealing with the operator Y_q has this property.)

§ 71. We shall now prove by induction that every Abelian group, with q unconnected operators, can be reduced, by a transformation of the variables, to the form

$$\frac{\partial}{\partial x_n}, \ \frac{\partial}{\partial x_{n-1}}, \ldots, \frac{\partial}{\partial x_{n-q+1}}.$$

Let $X_1, \ldots, X_q$ be the given operators of the group; then $X_1, \ldots, X_{q-1}$ will form a sub-group of $(q-1)$ unconnected Abelian operators. Assume that these can be reduced to the forms

$$\frac{\partial}{\partial x_{n-q+1}}, \ldots, \frac{\partial}{\partial x_{n-1}},$$

and that

$$X_q = \xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_{n-q} \frac{\partial}{\partial x_{n-q}} + \xi_{n-q+1} \frac{\partial}{\partial x_{n-q+1}} + \ldots + \xi_n \frac{\partial}{\partial x_n}.$$

The operators were unconnected and permutable in the first set of variables, and must therefore retain these properties in

the new variables; it then follows that none of the coefficients $\xi_1, \dots, \xi_n$ can contain $x_{n-q+1}, \dots, x_{n-1}$.

By a transformation of the form

$$y_1 = f_1(x_1, \dots, x_{n-q}, x_n), \dots, y_{n-q} = f_{n-q}(x_1, \dots, x_{n-q}, x_n),$$
$$y_n = f_n(x_1, \dots, x_{n-q}, x_n), \dots, y_{n-q+1} = x_{n-q+1}, \dots, y_{n-1} = x_{n-1},$$

we can, without altering the forms of $X_1, \dots, X_{q-1}$, reduce X_q to the form

$$\frac{\partial}{\partial y_n} + \xi_{n-q+1}\frac{\partial}{\partial y_{n-q+1}} + \dots + \xi_{n-1}\frac{\partial}{\partial y_{n-1}},$$

where $\xi_{n-q+1}, \dots, \xi_{n-1}$ are functions of $y_1, \dots, y_{n-q}, y_n$ only.

We may therefore suppose that $X_1, \dots, X_q$ have been thrown into the forms

$$X_q = \frac{\partial}{\partial x_n} + \xi_{n-q+1}\frac{\partial}{\partial x_{n-q+1}} + \dots + \xi_{n-1}\frac{\partial}{\partial x_{n-1}},$$
$$X_1 = \frac{\partial}{\partial x_{n-q+1}}, \dots, X_{q-1} = \frac{\partial}{\partial x_{n-1}},$$

where $\xi_{n-q+1}, \dots, \xi_{n-1}$ do not contain $x_{n-q+1}, \dots, x_{n-1}$; and to simplify the form of these operators further we take

$$y_1 = x_1, \dots, y_{n-q} = x_{n-q}, y_n = x_n,$$
$$y_{n-q+1} = x_{n-q+1} - \int \xi_{n-q+1}\, dx_n, \dots, y_{n-1} = x_{n-1} - \int \xi_{n-1}\, dx_n.$$

We now have

$$\frac{\partial}{\partial y_{n-q+1}} = \frac{\partial}{\partial x_{n-q+1}}, \dots, \frac{\partial}{\partial y_{n-1}} = \frac{\partial}{\partial x_{n-1}},$$
$$\frac{\partial}{\partial y_n} = \frac{\partial}{\partial x_n} + \xi_{n-q+1}\frac{\partial}{\partial x_{n-q+1}} + \dots + \xi_{n-1}\frac{\partial}{\partial x_{n-1}};$$

and therefore $X_1, \dots, X_q$ take the respective forms

$$\frac{\partial}{\partial y_{n-q+1}}, \frac{\partial}{\partial y_{n-q+2}}, \dots, \frac{\partial}{\partial y_n}.$$

As we have already proved that any single operator can be reduced to the form $\frac{\partial}{\partial y_n}$, we have now given an inductive proof that any q unconnected Abelian operators can, by a proper choice of variables, be reduced to the forms

$$\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-1}}, \dots, \frac{\partial}{\partial x_{n-q+1}}.$$

§ 72. When an Abelian group is reduced to this form $x_1, \dots, x_{n-q}$ are $(n-q)$ unconnected invariants of the group; and therefore we have proved that any complete system has exactly $(n-q)$ unconnected invariants.

It is important to prove that *these invariants can be obtained by direct algebraic processes and integrations of equations in* $(n-q+1)$ *variables at the most.*

To prove this we reduce the system to its normal form, which can be done by processes which are merely algebraic. If $X_1, \dots, X_q$ are now the operators we reduce X_q to the form $\frac{\partial}{\partial x_n}$; this we have proved can be done by quadratures, and the integration of an equation in $(n-q+1)$ variables at the most.

$X_1, \dots, X_{q-1}$ will now be $(q-1)$ unconnected Abelian operators; let

$$X_k = \xi_{k1}\frac{\partial}{\partial x_1} + \dots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \dots, q-1)$$

where, since X_k is permutable with $\frac{\partial}{\partial x_n}$, $\xi_{k1}, \dots, \xi_{kn}$ only involve $x_1, \dots, x_{n-1}$.

Our object being to obtain the invariants of $\frac{\partial}{\partial x_n}$ and $X_1, \dots, X_{q-1}$, it is only necessary to find those functions of $x_1, \dots, x_{n-1}$ which are annihilated by the $(q-1)$ linear operators

$$\xi_{k1}\frac{\partial}{\partial x_1} + \dots + \xi_{k\,n-1}\frac{\partial}{\partial x_{n-1}}, \qquad (k = 1, \dots, q-1).$$

These $(q-1)$ operators are Abelian operators, and unconnected, so that we have to find the invariants of an Abelian group in $(n-1)$ variables with $(q-1)$ unconnected operators.

Assuming then the theorem for the case of $(n-1)$ variables with $(q-1)$ operators, we see that it will also be true for the case of n variables with q operators; and since we have proved its truth when $q = 1$, we conclude that the *process of obtaining the common integrals of a complete system of linear partial differential equations, in n variables, involves the integration of linear equations in $(n-q+1)$ variables at the most.*

§ 73. Suppose now that we are given the equation

$$X_1(f) \equiv \xi_1\frac{\partial f}{\partial x_1} + \dots + \xi_n\frac{\partial f}{\partial x_n} = 0,$$

how far are we aided in finding its integrals by our knowledge

of $(q-1)$ other operators $X_2, \ldots, X_q$ forming with X_1 a complete system?

We first find the $(n-q)$ unconnected functions which are common integrals of

$$X_1(f) = 0, \ldots, X_q(f) = 0$$

by the method just explained; we then take these functions to form part of a new set of variables; and in these new variables may assume the integrals to be

$$x_n = a_n, \ldots, x_{q+1} = a_{q+1}.$$

We now have to find the remaining $(q-1)$ integrals of

$$(1) \qquad \xi_1 \frac{\partial f}{\partial x_1} + \ldots + \xi_q \frac{\partial f}{\partial x_q} = 0,$$

where $\xi_1, \ldots, \xi_q$ are functions of $x_1, \ldots, x_q, a_{q+1}, \ldots, a_n$; the subsidiary equations of (1) are then

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{dx_q}{\xi_q}.$$

It is known (Forsyth, *Differential Equations*, §§ 173, 174) that the solution of these subsidiary equations, and therefore of the corresponding linear partial differential equation (1), depends on the solution of an ordinary differential equation of order $(q-1)$ in one dependent, and one independent variable. Thus the solution of $\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} = 0$, where ξ and η are functions of x and y, depends on the solution of an ordinary equation of the first order; $\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \zeta \frac{\partial u}{\partial z} = 0$ depends on the solution of an ordinary differential equation of the second order.

If we define an *integration operation of order* m as the operation of obtaining the solution of an ordinary equation of order m, we may say that: *if we are given an equation $X_1(f) = 0$, and if we know $(q-1)$ other operators forming with X_1 a complete system of order q; the solution of the equation can be made to depend on algebraic processes, on quadratures, and on integration operations of order $(n-q)$ and $(q-1)$.*

Example. Prove that, if $X_1, \ldots, X_q$ is a complete system with the unconnected invariants $u_1, \ldots, u_{n-q}$, then every operator which annihilates each of these invariants is connected with $X_1, \ldots, X_q$.

By a change of the variables we may take the invariants

to be $x_1, \dots, x_{n-q}$; then the operators are in the variables $x_{n-q+1}, \dots, x_n$ only; and as they are unconnected

$$\frac{\partial}{\partial x_{n-q+1}}, \dots, \frac{\partial}{\partial x_n}$$

are each connected with $X_1, \dots, X_q$.

Any operator which annihilates $x_1, \dots, x_{n-q}$ must be of the form

$$\xi_{n-q+1}\frac{\partial}{\partial x_{n-q+1}} + \dots + \xi_n \frac{\partial}{\partial x_n},$$

and must therefore be connected with $X_1, \dots, X_q$.

CHAPTER VII

DIFFERENTIAL EQUATIONS ADMITTING KNOWN TRANSFORMATION GROUPS

§ 74. In this chapter we shall show how the fact, that a linear partial differential equation admits one or more infinitesimal transformations, which may be known by observation of the form of the equation or otherwise, enables us to reduce the order of the operations requisite for the solution of the given equation.

Let Y be the linear operator

$$\eta_1 \frac{\partial}{\partial x_1} + \ldots + \eta_n \frac{\partial}{\partial x_n},$$

where $\eta_1, \ldots, \eta_n$ are functions of $x_1, \ldots, x_n$, and Y' the operator obtained from Y by replacing x_i by x'_i.

(1) If $\qquad x'_i = e^{tX} x_i, \qquad (i = 1, \ldots, n),$

where $\qquad X = \xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_n \frac{\partial}{\partial x_n},$

we must obtain an expression for Y' in terms of $x_1, \ldots, x_n$, and this will enable us to determine at once if the equation $Y(f) = 0$ admits the transformation (1).

From (1) we deduce (§ 44) $x_i = e^{-tX'} x'_i$ and therefore $Y'x_i = Y'e^{-tX'} x'_i$. Since $Y'e^{-tX'} x'_i$ is a function of $x'_1, \ldots, x'_n$ we therefore have

$$(2) \qquad Y' x_i = e^{tX} Y e^{-tX} x_i.$$

Expanding $e^{tX} Y e^{-tX}$ in powers of t, we see that the coefficient of t^r is

$$\frac{X^r Y}{r!} - \frac{X^{r-1} YX}{(r-1)!} + \frac{X^{r-2} YX^2}{(r-2)!\, 2!} - \frac{X^{r-3} YX^3}{(r-3)!\, 3!} + \ldots.$$

We shall prove that this expression is equal to $(-1)^r \dfrac{Y^{(r)}}{r!}$ where $\quad Y^{(1)} = YX - XY, \quad Y^{(2)} = Y^{(1)} X - XY^{(1)}, \ldots,$

$$Y^{(r)} = Y^{(r-1)} X - XY^{(r-1)},$$

$Y^{(r)}$ having now the meaning which was attached to y_r in § 48.

Assume that

$$(-1)^{r-1}\frac{Y^{(r-1)}}{(r-1)!}=\frac{X^{r-1}Y}{(r-1)!}-\frac{X^{r-2}YX}{(r-2)!\,1!}+\frac{X^{r-3}YX^2}{(r-3)!\,2!}-\ldots$$

then $$\frac{(-1)^r(Y^{(r-1)}X-XY^{(r-1)})}{(r-1)!}$$

$$=\frac{X^rY}{(r-1)!}-(r-1)\frac{X^{r-1}YX}{(r-1)!\,1!}+(r-2)\frac{X^{r-2}YX^2}{(r-2)!\,2!}-\ldots$$

$$-\frac{X^{r-1}YX}{(r-1)!\,1!}+2\frac{X^{r-2}YX^2}{(r-2)!\,2!}-\ldots,$$

$$=\frac{X^rY}{(r-1)!}-r\frac{X^{r-1}YX}{(r-1)!\,1!}+r\frac{X^{r-2}YX^2}{(r-2)!\,2!}-\ldots,$$

and therefore

$$(-1)^r\frac{Y^{(r)}}{r!}=\frac{X^rY}{r!}-\frac{X^{r-1}YX}{(r-1)!\,1!}+\frac{X^{r-2}YX^2}{(r-2)!\,2!}-\ldots,$$

so that the required theorem is proved by induction; and

$$e^{tX}Ye^{-tX}=Y-tY^{(1)}+\frac{t^2}{2!}Y^{(2)}-\frac{t^3}{3!}Y^{(3)}+\ldots.$$

It follows that $e^{tX}Ye^{-tX}$ is a linear operator, and as such it may be written in the form

$$\sum^{i=n}(e^{tX}Ye^{-tX}x_i)\frac{\partial}{\partial x_i},$$

and by (2) this may be written

$$\sum^{i=n}Y'(x_i)\frac{\partial}{\partial x_i}=Y',$$

so that $$Y'=Y-\frac{t}{1!}Y^{(1)}+\frac{t^2}{2!}Y^{(2)}-\frac{t^3}{3!}Y^{(3)}+\ldots.$$

§ 75. We may apply this formula to obtain the conditions that a given sub-group may be self-conjugate.

If $X_1, \ldots, X_n$ are the infinitesimal operators of a group, of which $X_{q+1}, \ldots, X_r$ form a sub-group, we defined a self-conjugate sub-group as one such that

$$e^{e_1X_1+\ldots+e_rX_r}e^{\lambda_{q+1}X_{q+1}+\ldots+\lambda_rX_r}e^{-e_1X_1-\ldots-e_rX_r}$$

is always an operation of the sub-group, whatever be the

values of $e_1, \ldots, e_r$, the parameters of the group, or $\lambda_{q+1}, \ldots, \lambda_r$ the parameters of the sub-group.

If we denote by X the operator $e_1 X_1 + \ldots + e_r X_r$, this condition may be expressed by saying that the group generated by $X'_{q+1}, \ldots, X'_r$, where

$$x'_i = e^X x_i,$$

is identical with the group $X_{q+1}, \ldots, X_r$; that is, that each operator $X'_{q+1}, \ldots, X'_r$ is dependent on the operators of the set $X_{q+1}, \ldots, X_r$.

Now the formula we have just proved gives

$$X'_k = X_k - X_k^{(1)} + \frac{1}{2!} X_k^{(2)} - \frac{1}{3!} X_k^{(3)} + \ldots, \qquad (k = q+1, \ldots, r),$$

so that

$$X_{q+j}^{(1)} - \frac{1}{2!} X_{q+j}^{(2)} + \frac{1}{3!} X_{q+j}^{(3)} - \ldots, \qquad (j = 1, \ldots, r-q)$$

must be dependent on $X_{q+1}, \ldots, X_r$.

By the second fundamental theorem (§ 47) we have

$$X_{q+j}^{(1)} = \sum^{i=k=r} e_i c_{q+j,\, i,\, k} X_k,$$

and therefore, if we take $e_1, \ldots, e_r$ so small that their squares may be neglected, we see that a necessary condition for $X'_{q+1}, \ldots, X'_r$ being dependent on $X_{q+1}, \ldots, X_r$ is

$$\sum^{i=r} e_i c_{q+j,\, i,\, k} = 0, \qquad (k = 1, \ldots, q).$$

Since this must be true whatever the values of the small quantities $e_1, \ldots, e_r$ we must have

$$c_{q+j,\, i,\, k} = 0, \qquad \left(\begin{matrix} j = 1, \ldots, r-q; \\ i = 1, \ldots, r; \end{matrix} \quad k = 1, \ldots, q\right).$$

The sub-group $X_{q+1}, \ldots, X_r$ cannot then be self-conjugate unless these conditions are satisfied.

These necessary relations between the structure-constants are also sufficient; for if they are satisfied $X_{q+j}^{(1)}$ will be dependent on $X_{q+1}, \ldots, X_r$; and therefore, since this is true for all values of j from 1 to $r-q$, $X_{q+j}^{(2)}, X_{q+j}^{(3)}, \ldots$ will all be dependent on $X_{q+1}, \ldots, X_r$, and therefore X'_{q+j} will be so dependent.

If we take $q = r-1$, we get in particular as the conditions that X_r may be a self-conjugate operator

$$c_{rik} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r \\ k = 1, \ldots, r-1 \end{pmatrix}.$$

If X_r is to be an Abelian operator the further conditions

$$c_{rir} = 0, \qquad (i = 1, \ldots, r)$$

are necessary.

§ 76. We now seek the conditions that the complete system of equations

$$Y_1(f) = 0, \ldots, Y_q(f) = 0$$

may admit the group of order one

$$x_i' = e^{tX} x_i, \qquad (i = 1, \ldots, n).$$

Clearly the conditions are that $Y_1', \ldots, Y_q'$ should each be connected with $Y_1, \ldots, Y_q$; that is, we must have

$$Y_k' = \rho_{k1} Y_1 + \ldots + \rho_{kq} Y_q, \qquad (k = 1, \ldots, q),$$

where $\rho_{ki}, \ldots$ are functions of $x_1, \ldots, x_n$.

Since
$$Y_k' = Y_k - tY_k^{(1)} + \frac{t^2}{2!} Y_k^{(2)} - \ldots,$$

we see, by taking t very small, that necessary conditions are

$$Y_k^{(1)} = \sigma_{k1} Y_1 + \ldots + \sigma_{kq} Y_q, \qquad (k = 1, \ldots, q),$$

where $\sigma_{ki}, \ldots$ are some functions of $x_1, \ldots, x_n$.

These necessary conditions are also sufficient; for

$$\begin{aligned} Y_k^{(2)} = (\sigma_{k1} Y_1 + \ldots + \sigma_{kq} Y_q, X) &= \sigma_{k1} Y_1^{(1)} + \ldots + \sigma_{kq} Y_q^{(1)} \\ &\quad + (X\sigma_{k1}) Y_1 + \ldots + (X\sigma_{kq}) Y_q, \end{aligned}$$

and therefore, since $Y_1^{(1)}, \ldots, Y_q^{(1)}$ are each connected with $Y_1, \ldots, Y_q$, we see that $Y_k^{(2)}$ is also connected with $Y_1, \ldots, Y_q$.

Similarly we see that $Y_k^{(3)}$, $Y_k^{(4)}, \ldots$ are each so connected; and therefore $Y_1', \ldots, Y_q'$ are connected with $Y_1, \ldots, Y_q$; and we conclude that *the necessary and sufficient conditions that a complete system of linear partial differential equations of the first order should admit the group*

$$x_i' = e^{tX} x_i, \qquad (i = 1, \ldots, n)$$

are that the alternants $(Y_1, X), \ldots, (Y_q, X)$ *should each be connected with* $Y_1, \ldots, Y_q$.

§ 77. If $f(x_1, \ldots, x_n) =$ constant is any integral of the complete system, that is, if $f(x_1, \ldots, x_n)$ is any invariant of the complete system of operators $Y_1, \ldots, Y_q$, then $f(x'_1, \ldots, x'_n)$ is an invariant of $Y'_1, \ldots, Y'_q$. Now by hypothesis the complete system admits

$$x'_i = e^{tX} x_i,$$

and therefore by what we have just proved

$$Y'_k = \rho_{k1} Y_1 + \ldots + \rho_{kq} Y_q, \qquad (k = 1, \ldots, q).$$

The determinant of the functions $\rho_{ik}, \ldots$ cannot be zero; for if it were zero $Y'_1, \ldots, Y'_q$ would be connected, and there- $Y_1, \ldots, Y_q$ (being operators of the same form, but in the variables $x_1, \ldots, x_n$ instead of $x'_1, \ldots, x'_n$) would be connected, and this is contrary to hypothesis: since then the determinant is not zero, every invariant of $Y'_1, \ldots, Y'_q$ is an invariant of $Y_1, \ldots, Y_q$; and we conclude that if $f(x_1, \ldots, x_n)$ is an invariant of $Y_1, \ldots, Y_q$ so also is $f(x'_1, \ldots, x'_n)$.

In other words, *any invariant of the complete system of operators is transformed by*

$$x'_i = e^{tX} x_i, \qquad (i = 1, \ldots, n)$$

into some other invariant function, if the complete system admits this transformation.

We may prove conversely that *if*

$$x'_i = e^{tX} x_i, \qquad (i = 1, \ldots, n)$$

transforms every invariant of the complete system into some other invariant, then the complete system admits this transformation.

For suppose that $f(x_1, \ldots, x_n)$ is an invariant: then by the hypothesis so is $f(x'_1, \ldots, x'_n)$, that is

$$e^{tX} f(x_1, \ldots, x_n)$$

is an invariant. If we now take t very small, we may conclude that $Xf(x_1, \ldots, x_n)$ is an invariant, and therefore must be annihilated by $Y_1, \ldots, Y_q$.

Since $f(x_1, \ldots, x_n)$ is an invariant, it is annihilated by $Y_1, \ldots, Y_q$, and therefore also by the operators of the second degree $XY_1, \ldots XY_q$; and therefore finally $f(x_1, \ldots, x_n)$ is annihilated by each of the alternants $(Y_1, X), \ldots, (Y_q, X)$.

It follows then from the example on page 89 that each of these alternants is connected with $Y_1, \ldots, Y_q$, and therefore that the complete system admits

$$x'_i = e^{tX} x_i, \qquad (i = 1, \ldots, n).$$

We thus see that the conditions that a complete system may admit the above group may be expressed by either of two equivalent conditions; firstly, by the condition that the alternants of each of the operators of the complete system with X should be *connected* with the operators of this system; or, secondly, by the condition that every invariant of the system should be transformed into another invariant by the operator X.

§ 78. The condition that a given function $f(x_1, \ldots, x_n)$ may admit

(1) $\quad x_i' = x_i + t\,\xi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$

is that it should be annihilated by the operator X,

where $$X = \xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_n \frac{\partial}{\partial x_n}.$$

It must therefore, if it admits (1), also admit

(2) $\quad x_i' = x_i + t\rho\,\xi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$

whatever function of the variables $x_1, \ldots, x_n$ the multiplier ρ may be.

If on the other hand a given *differential equation* $Y(f) = 0$ admits (1), it will not in general admit (2).

If $Y_1(f) = 0, \ldots, Y_q(f) = 0$ is a given complete system of differential equations the system will obviously admit the infinitesimal transformation.

(3) $$x_i' = x_i + t\,(\rho_1 Y_1 + \ldots + \rho_q Y_q)\, x_i$$

whatever the functions $\rho_1, \ldots, \rho_q$ may be; for the alternants of $Y_1, \ldots, Y_q$ with $\rho_1 Y_1 + \ldots + \rho_q Y_q$ are connected with $Y_1, \ldots, Y_q$.

A transformation of the form (3) is said to be *trivial*.

If the equation system admits

$$x_i' = e^{tX} x_i,$$

we say that it admits the operator X; and we now see that if it admits X it will also admit

$$X + \rho_1 Y_1 + \ldots + \rho_q Y_q;$$

but with respect to the given equation system we should not

reckon $$x_i' = e^{tX} x_i$$

and $$x_i' = e^{tX + \rho_1 Y_1 + \ldots + \rho_q Y_q} x_i$$

as *distinct* transformations.

We can, however, make use of the fact that $\rho_1, \dots, \rho_q$ are undetermined to obtain the simplest forms of the operators admitted by the given equation system.

Suppose that the complete system admits the non-trivial transformation
$$x_i' = x_i + t\,\xi_i(x_1, \dots, x_n),$$
under what conditions will it admit
$$x_i' = x_i + t\rho\,\xi_i(x_1, \dots, x_n)\,?$$

The conditions are that the alternants $(Y_1, \rho X), \dots, (Y_q, \rho X)$ should each be connected with $Y_1, \dots, Y_q$; and therefore, since $\rho(Y_1, X), \dots, \rho(Y_q, X)$ are each so connected,
$$(Y_1\rho)X, \dots, (Y_q\rho)X$$
must each be connected with $Y_1, \dots, Y_q$.

Now by hypothesis X is not connected with $Y_1, \dots, Y_q$; and therefore we must have
$$Y_1\rho = 0, \dots, Y_q\rho = 0;$$
that is ρ is either a constant, or an invariant of the complete system.

§ 79. If the complete system is reduced to *normal form*, that is if
$$Y_k = \frac{\partial}{\partial x_{n-q+k}} + \sum^{i=n-q} \eta_{ki}\frac{\partial}{\partial x_i}, \qquad (k = 1, \dots, q),$$
the further discussion of the problem with which we are now concerned is made more simple. This problem is the investigation of the reduction of the order of the integration operations, necessary for the solution of the given equation system, due to the fact that the system admits known non-trivial transformations.

Since the reduction of the system to normal form only involves algebraic processes, we may suppose the system to be given in normal form.

If X is a non-trivial operator admitted by the system, then
$$X + \rho_1 Y_1 + \dots + \rho_q Y_q$$
is also admitted, and is non-trivial; and, by properly choosing the functions $\rho_1, \dots, \rho_q$, we can replace X by a linear operator of the form
$$\xi_1\frac{\partial}{\partial x_1} + \dots + \xi_{n-q}\frac{\partial}{\partial x_{n-q}}$$
which is necessarily non-trivial.

We shall call such an operator a *reduced* operator; and when we are given any non-trivial operator admitted by the system, we replace it—and this can be done by mere algebra—by the corresponding *reduced* operator.

If then we are given a complete system, in normal form, admitting m known unconnected reduced operators $X_1, \ldots, X_m$ we must have

$$(X_i, Y_k) = \sigma_1 Y_1 + \ldots + \sigma_q Y_q.$$

Now in (X_i, Y_k) the coefficients of $\frac{\partial}{\partial x_{n-q+1}}, \ldots, \frac{\partial}{\partial x_n}$ are all zero, and therefore we must have $\sigma_1 = 0, \ldots, \sigma_q = 0$; each of the operators $X_1, \ldots, X_m$ is therefore permutable with each of the operators $Y_1, \ldots, Y_q$. Also there cannot be more than $(n-q)$ reduced unconnected operators $X_1, \ldots, X_m$, for these operators are in the $(n-q)$ variables $x_1, \ldots, x_{n-q}$ only, $x_{n-q+1}, \ldots, x_n$ entering them merely as parameters.

We also see as in § 78 that

$$\rho_1 X_1 + \ldots + \rho_m X_m$$

can only be admitted if $\rho_1, \ldots, \rho_m$ are invariants of the operators $Y_1, \ldots, Y_q$.

From the Jacobian identity

$$(Y_k, (X_i, X_j)) + (X_j, (Y_k, X_i)) + (X_i, (X_j, Y_k)) \equiv 0,$$

we see that, since (Y_k, X_i) and (Y_k, X_j) vanish identically, so also must $(Y_k, (X_i, X_j))$; that is, the equation system admits the alternant of any two reduced operators; and this alternant is itself a reduced operator since it is of the form

$$\xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_{n-q} \frac{\partial}{\partial x_{n-q}}.$$

It therefore follows that, if an equation system admits any non-trivial operators at all, it must admit a *complete* system of operators; we shall suppose then that $X_1, \ldots, X_m$ is a complete system of operators in the variables $x_1, \ldots, x_{n-q}$, the other variable $x_{n-q+1}, \ldots, x_n$ entering these operators only as parameters; and we know that $m \not> n-q$.

§ 80. We now have

$$(X_i, X_j) = \rho_{ij1}, X_1 + \ldots + \rho_{ijm} X_m,$$

and, since the system admits (X_i, X_j), the functions $\rho_{ijk}, \ldots$ are either constants, or integrals of the given equation system.

The first thing which we must now do is to reduce the case where the functions are integrals to the case where they are mere constants.

Suppose that of the functions $\rho_{ijk}, \ldots$ exactly s are unconnected; we now know s invariants of the complete system, and we therefore transform to a new set of variables, so chosen that $x_{n-q}, x_{n-q+1}, \ldots, x_{n-q-s+1}$ are these known invariants of the complete system.

This transformation of the variables has only involved algebraic processes; and we now again bring the system to normal form, when we have

$$Y_k = \frac{\partial}{\partial x_{n-q+k}} + \sum^{i=n-q-s} \eta_{ki} \frac{\partial}{\partial x_i}, \qquad (k = 1, \ldots, q).$$

We suppose $X_1, \ldots, X_m$, the operators which the equation system admits, again *reduced*, so that

$$X_k = \sum^{i=n-q} \xi_{ki} \frac{\partial}{\partial x_i}, \qquad (k = 1, \ldots, m).$$

From the fact that $(Y_i, X_k) = 0$, and that none of the terms $\frac{\partial}{\partial x_{n-q-s+1}}, \ldots, \frac{\partial}{\partial x_{n-q}}$ occur in $Y_1, \ldots, Y_q$, we see that

$$Y_j \xi_{kh}, = 0, \qquad \left(\begin{matrix} h = n-q-s+1, \ldots, n-q \\ j = 1, \ldots, q \end{matrix}\right).$$

It therefore follows that $\xi_{kh}, \ldots$ are integrals of the system: they may either be new integrals or they may be connected with the known set $x_{n-q}, \ldots, x_{n-q-s+1}$.

If they are new integrals we simplify $Y_1, \ldots, Y_q$ still further by again introducing the new integrals as variables; and continue to do this till we can obtain no further integrals by this method.

We may therefore now assume that

$$\xi_{kh}, \ldots, \qquad (h = n-q-s+1, \ldots, n-q)$$

are merely functions of $x_{n-q}, \ldots, x_{n-q-s+1}$, that is, of the integrals already known.

§ 81. It must be noticed that we cannot advance further in obtaining integrals of the complete system, through our knowledge that the system admits $X_1, \ldots, X_m$, *unless in so*

far as we know how to deduce from $X_1, \dots, X_m$ *operators of the form*

$$\sum^{i=n-q-s} \xi_{ki} \frac{\partial}{\partial x_i}.$$

To prove this, suppose that the system admits X which is of the form

$$\frac{\partial}{\partial x_{n-q-s+1}} + \sum^{i=n-q-s} \xi_{1i} \frac{\partial}{\partial x_i}.$$

We now have the complete system of equations

$$X(f) = 0, Y_1(f) = 0, \dots, Y_q(f) = 0,$$

and it is in normal form; but, since we have increased the number of the variables as well as of the equations, the order of the integration operations, necessary to find a common integral, is now no lower than it was to find a common integral of

$$Y_1(f) = 0, \dots, Y_q(f) = 0.$$

We take

$$Z_k = \rho_{ki} X_1 + \dots + \rho_{km} X_m, \qquad (k = 1, \dots, m),$$

where $\rho_{ki}, \dots$ are functions of $x_{n-q}, \dots, x_{n-q-s+1}$ only, and are therefore invariants of $Y_1, \dots, Y_q$. $Z_1, \dots, Z_m$ will now be reduced operators admitted by the given equation system.

We must so choose $\rho_{ki}, \dots$ as to obtain as many as possible of the operators in the form

$$\xi_1 \frac{\partial}{\partial x_1} + \dots + \xi_{n-q-s} \frac{\partial}{\partial x_{n-q-s}},$$

and these alone can be effective for our purpose.

§ 82. The problem before us is now simplified and may be thus restated: we are given q operators $Y_1, \dots, Y_q$ where

$$Y_k = \frac{\partial}{\partial x_{n-q+k}} + \sum^{i=n-q-s} \eta_{ki} \frac{\partial}{\partial x_i}, \qquad (k = 1, \dots, q);$$

and, in order to obtain new integrals of the system, we are to make the most use of our knowledge that the system admits $X_1, \dots, X_m$ where

$$X_k = \sum^{i=n-q-s} \xi_{ki} \frac{\partial}{\partial x_i}, \qquad (k = 1, \dots, m).$$

As before we have

$$(X_i, X_j) = \rho_{ij1} X_1 + \ldots + \rho_{ijm} X_m,$$

and the functions $\rho_{ijk}, \ldots$ being invariants, we should have new integrals unless they are merely functions of the known integrals $x_{n-q}, \ldots, x_{n-q-s+1}$.

Since we have assumed that we cannot obtain any more integrals by this method we must take these quantities $\rho_{ijk}, \ldots$ to be merely functions of $x_{n-q}, \ldots, x_{n-q-s+1}$; and, since these variables only enter $Y_1, \ldots, Y_q, X_1, \ldots, X_m$ as parameters, we may now assume $\rho_{ijk}, \ldots$ to be mere constants.

The operators $X_1, \ldots, X_m$ then satisfy the identities

$$(X_i, X_j) = \sum^{k=m} c_{ijk} X_k, \qquad \binom{i = 1, \ldots, m}{j = 1, \ldots, m},$$

that is, they generate a group.

We thus see how Lie's theory of finite continuous groups had its origin in the question which he proposed, viz. what advance can be made towards the solution of linear partial differential equations of the first order, by the knowledge of the infinitesimal transformations which the equation admits?

§ 83. We know that $(m+q)$ is not greater than n; suppose that it is less than n. We then find the common integrals of the complete system

$$X_1(f) = 0, \ldots, X_m(f) = 0, \quad Y_1(f) = 0, \ldots, Y_q(f) = 0,$$

of which all the operators are unconnected, and of which the structure of the operators—for these operators generate a group of order $(m+q)$—is given by

$$(X_i, X_k) = c_{ik1} X_1 + \ldots + c_{ikm} X_m,$$

and by the fact that the operators $Y_1, \ldots, Y_q$ are Abelian operators within the group of order $m+q$.

There are $(n-m-q)$ common integrals of this system which can be found by an integration operation of order $(n-m-q)$. Having determined these integrals we so change the variables that the corresponding invariant functions become

$$x_n, \ldots, x_{m+q+1};$$

and the problem of finding the remaining integrals of

$$Y_1(f) = 0, \ldots, Y_q(f) = 0$$

is now reduced to that of finding the invariants of a complete system of order q, in $(m+q)$ variables $x_1, \ldots, x_{m+q}$, the system admitting m known reduced unconnected operators, also in the same variables $x_1, \ldots, x_{m+q}$.

As $(m+q)$ is either less than n or equal to it, we can now restate the problem in the form to which we have reduced it.

Given a complete system of equations

$$Y_1(f) = 0, \ldots, Y_q(f) = 0$$

in $(r+q)$ variables $x_1, \ldots, x_{r+q}$, whose invariants are required, we are to take advantage of the fact that the system admits r known operators $X_1, \ldots, X_r$ in these variables.

The r operators are unconnected, and reduced, and generate a group which is finite and continuous; and the variables $x_n, \ldots, x_{n-r-q+1}$ occur in $X_1, \ldots, X_r$, $Y_1, \ldots, Y_q$, merely as parameters; $Y_1, \ldots, Y_q$ are operators permutable with each other and with $X_1, \ldots, X_r$.

§ 84. In order to find the invariants of $Y_1, \ldots, Y_q$ we should have required integration operations of order r, had it not been that we know that the equation system admits the operators $X_1, \ldots, X_r$. We therefore find the maximum sub-group of $X_1, \ldots, X_r$; that is, the sub-group with the greatest number of independent operators, which being a sub-group must not include all the operators of the given group $X_1, \ldots, X_r$; and we find the integrals of the system

$$Y_1(f) = 0, \ldots, Y_q(f) = 0, \quad X_1(f) = 0, \ldots, X_m(f) = 0,$$

where $X_1, \ldots, X_m$ is this maximum sub-group.

To obtain these integrals, integration operations of order $(r-m)$ are required, and $(r-m)$ integrals are thus obtained; the reason why we choose m as large as possible is to reduce the order of the necessary operations; and the reason why we choose a sub-group is to ensure that $(r-m)$ shall not vanish.

We shall now show how, by merely algebraic processes, we may obtain other integrals from these $(r-m)$ integrals.

§ 85. The principle which enables us to find these additional integrals is that explained in § 77. Since the given system admits $X_1, \ldots, X_r$, we know that if $\phi(x_1, \ldots, x_n)$ is any invariant of $Y_1, \ldots, Y_q$, then $X_1\phi, \ldots, X_r\phi$ will also be invariants. All of the invariants we have already found can be annihilated by $X_1, \ldots, X_m$; but they cannot all be annihilated by X_{m+1}, nor by any of the operators $X_{m+2}, \ldots, X_r$; we

may therefore by this method be enabled to obtain new integrals.

By a change of the variables, that is, by an algebraic process, we may take the invariants already known to be $x_{r+q}, \ldots, x_{q+m+1}$.

Let $X_1, \ldots, X_l$ be that maximum sub-group of $X_1, \ldots, X_m$ which is self-conjugate within $X_1, \ldots, X_r$; if $X_1, \ldots, X_m$ is itself self-conjugate within $X_1, \ldots, X_r$, we may take $X_1, \ldots, X_l$ to be the sub-group $X_1, \ldots, X_m$ itself.

The proposition which we are now going to establish is this—*by operating with* $X_1, \ldots, X_r$ *on the known invariants* $x_{r+q}, \ldots, x_{q+m+1}$ *we obtain the common integrals of*

$$Y_1(f) = 0, \ldots, Y_q(f) = 0, \quad X_1(f) = 0, \ldots, X_l(f) = 0;$$

that is, we obtain exactly $(m-l)$ *additional integrals.*

Since all of the variables $x_{r+q}, \ldots, x_{q+m+1}$ are invariants of $Y_1, \ldots, Y_q, X_1, \ldots, X_m$ they must also be invariants of $Y_1, \ldots, Y_q, X_1, \ldots, X_l$; by a change of variables we may take $x_{q+m}, \ldots, x_{q+l+1}$ to be the remaining invariants of

$$Y_1, \ldots, Y_q, \ X_1, \ldots, X_l;$$

we are now about to prove that by performing known operations on $x_{r+q}, \ldots, x_{m+q+1}$ we must obtain these additional invariants.

Since $X_1, \ldots, X_l$ is a self-conjugate sub-group of $X_1, \ldots X_r$, the equations

$$X_1(f) = 0, \ldots, X_l(f) = 0, \quad Y_1(f) = 0, \ldots, Y_q(f) = 0$$

admit the operators $X_1, \ldots, X_r$; and therefore the functions obtained by operating with $X_1, \ldots, X_r$ on $x_{r+q}, \ldots, x_{m+q+1}$ must all be invariants of $X_1, \ldots, X_l, Y_1, \ldots, Y_q$.

Now $X_1, \ldots, X_l, Y_1, \ldots, Y_q$ are unconnected, and have as invariants the $(r-l)$ variables $x_{q+l+1}, \ldots, x_{r+q}$; every other invariant must therefore be a function of these variables only; and therefore we know that the invariants obtained by operating with $X_1, \ldots, X_r$ are functions of $x_{q+l+1}, \ldots, x_{r+q}$ only.

If $(r-l)$ of these invariants are unconnected, then

$$x_{q+l+1}, \ldots, x_{r+q}$$

can be expressed in terms of these invariants; but if fewer than $(r-l)$ of the invariants are unconnected, they cannot be so expressed; and we therefore know that there must be some operator of the form

$$\xi_{q+l+1}\frac{\partial}{\partial x_{q+l+1}} + \ldots + \xi_{q+m}\frac{\partial}{\partial x_{q+m}}$$

which annihilates each of the functions $X_i x_{q+m+j}$, where j may have any value from 1 to $(r-m)$, and i any value from 1 to r, and where $\xi_{q+l+1}, \ldots, \xi_{q+m}$ are not all zero.

Since $x_{q+m+1}, \ldots, x_{r+q}$ are invariants of

$$Y_1, \ldots, Y_q,\ X_1, \ldots, X_m,$$

and these operators are unconnected, we see that

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{q+m}}$$

must be connected with $Y_1, \ldots, Y_q,\ X_1, \ldots, X_m$; we can therefore replace

$$\xi_{q+l+1}\frac{\partial}{\partial x_{q+l+1}} + \ldots + \xi_{q+m}\frac{\partial}{\partial x_{q+m}}$$

by an operator of the form

$$(1) \qquad \rho_1 Y_1 + \ldots + \rho_q Y_q + \sigma_1 X_1 + \ldots + \sigma_m X_m,$$

where $\rho_1, \ldots, \rho_q,\ \sigma_1, \ldots, \sigma_m$, are functions of the variables.

Now each of the operators $Y_1, \ldots, Y_q, X_1, \ldots, X_m$ annihilates each of the variables $x_{q+m+1}, \ldots, x_{r+q}$, and (1) annihilates any function $X_i x_{q+m+j}$; we conclude then that

$$\rho_1(Y_1, X_i) + \ldots + \rho_q(Y_q, X_i) + \sigma_1(X_1, X_i) + \ldots + \sigma_m(X_m, X_i)$$

annihilates each of the variables $x_{q+m+1}, \ldots, x_{r+q}$.

From the known relations between the alternants of the operators $Y_1, \ldots, Y_q,\ X_1, \ldots, X_m$ we see that

$$\sum^{j=m,\, k=r} \sigma_j c_{jik} X_k, \qquad (i = 1, \ldots, r)$$

annihilates each of these variables; and must therefore be connected with the operators of which $x_{q+m+1}, \ldots, x_{r+q}$ are the invariants; that is, with $Y_1, \ldots, Y_q,\ X_1, \ldots, X_m$.

It follows that, these operators being all *unconnected*, we must have

$$\sum^{j=m} \sigma_j c_{jik} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r \\ k = m+1, \ldots, r \end{pmatrix}.$$

Now because $X_1, \ldots, X_l$ is a self-conjugate sub-group

$$c_{jik} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r; & \\ j = 1, \ldots, l; & k = l+1, \ldots, r \end{pmatrix};$$

and therefore

$$\sum_{j=l+1}^{j=m} \sigma_j c_{jik} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r \\ k = m+1, \ldots, r \end{pmatrix}.$$

If $\quad c_{jik} = 0, \quad \left(\begin{matrix} i = 1, \ldots, r; \\ j = l+1, \ldots, m; \end{matrix} \quad k = m+1, \ldots, r\right),$

$X_1, \ldots, X_m$ is a self-conjugate sub-group, and $l = m$. If $m > l$, these constants cannot all vanish (for then the greatest sub-group would be of order $> l$); and we can take one of the functions $\sigma_{l+1}, \ldots, \sigma_m$ to be *dependent* on the others; it follows that without altering the structure of $X_1, \ldots, X_l$, or without transforming the sub-group $X_1, \ldots, X_m$ into any other sub-group, we may choose instead of $X_{l+1}, \ldots, X_m$ certain $(m-l)$ independent operators which will be dependent on $X_{l+1}, \ldots, X_m$, and for this new set we may take σ_m to be zero.

If we now consider the corresponding new structure constants, we shall as before obtain the identities of the form

$$\sum_{j=l+1}^{j=m-1} \sigma_j c_{jik} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r \\ k = m+1, \ldots, r \end{matrix}\right);$$

and can similarly choose σ_{m-1} to be zero, and, proceeding thus, finally cause all the functions $\sigma_{l+1}, \ldots, \sigma_m$ to disappear.

It would then follow that

$$\xi_{q+l+1} \frac{\partial}{\partial x_{q+l+1}} + \ldots + \xi_{q+m} \frac{\partial}{\partial x_{q+m}}$$

could be replaced by an operator of the form

$$(2) \qquad \rho_1 Y_1 + \ldots + \rho_q Y_q + \sigma_1 X_1 + \ldots + \sigma_l X_l;$$

but this is impossible since (2) annihilates $x_{q+l+1}, \ldots, x_{q+m}$: we must therefore draw the conclusion that $x_{q+l+1}, \ldots, x_{q+m}$ can be expressed in terms of the invariants obtained by operating on the known invariants $x_{q+m+1}, \ldots, x_{r+q}$ with $X_{m+1}, \ldots, X_r$.

§ 86. It therefore follows from what we have proved that we can by an integration operation of order $(r-m)$ obtain $(r-l)$ invariants of $Y_1, \ldots, Y_q$; and we may take these to be $x_{r+q}, \ldots, x_{q+l+1}$, by a transformation to new variables.

The variables $x_{r+q}, \ldots, x_{q+l+1}$ now appear only as parameters in $Y_1, \ldots, Y_q$; we can therefore, by processes which are merely algebraic, select from the r operators $X_1, \ldots, X_r$ which the equation system admits l operators, in which also

$$x_{r+q}, \ldots, x_{q+l+1}$$

will only appear as parameters. These will form a group of

order l in $(l+q)$ variables, and will be unconnected with one another, or with $Y_1, \ldots, Y_q$. The equation system

$$Y_1(f) = 0, \ldots, Y_q(f) = 0$$

will admit these operators, and the problem which is now before us is exactly the same as it was before, but we have only $(l+q)$ variables to deal with, whereas before we had $(r+q)$.

§ 87. There is one case of special interest in this general theory, viz. when the greatest sub-group of $X_1, \ldots, X_r$ is self-conjugate.

Since $X_1, \ldots, X_m$ is self-conjugate, the alternant of any of these operators with X_{m+1} is dependent on $X_1, \ldots, X_m$; and therefore $X_1, \ldots, X_{m+1}$ is itself a sub-group; but $X_1, \ldots, X_m$ is by hypothesis the maximum sub-group, and therefore $X_1, \ldots, X_{m+1}$ must be the group $X_1, \ldots, X_r$ itself.

When the greatest sub-group of $X_1, \ldots, X_r$ is self-conjugate its order must therefore be $(r-1)$.

There is only one invariant of $Y_1, \ldots, Y_q, X_1, \ldots, X_{r-1}$; suppose it to be $f(x_1, \ldots, x_{r+q})$, then, since $X_r(f)$ must also be an invariant,

$$X_r f(x_1, \ldots, x_{r+q}) = F\big(f(x_1, \ldots, x_{r+q})\big),$$

where F is some functional symbol.

This function $F\big(f(x_1, \ldots, x_{r+q})\big)$ cannot be zero; for

$$Y_1, \ldots, Y_q, \; X_1, \ldots, X_r$$

being unconnected have no common invariant; there must therefore be some function of $f(x_1, \ldots, x_{r+q})$, such that, when operated on by X_r, the result will be unity.

Let u be this required function, then

$$Y_1(u) = 0, \ldots, Y_q(u) = 0, \; X_1(u) = 0, \ldots, X_{r-1}(u) = 0,$$
$$X_r(u) = 1.$$

Since these are $(r+q)$ unconnected equations in $(r+q)$ variables every derivative of u is known; that is, $\dfrac{\partial u}{\partial x_i}, \ldots, \dfrac{\partial u}{\partial x_{q+r}}$ are each known, and u can therefore be obtained by mere quadrature. By transforming to a new set of variables we may take this function to be x_{q+r}; since x_{q+r} will then occur merely as a parameter in $Y_1, \ldots, Y_q, X_1, \ldots, X_{r-1}$ we shall then be given an equation system

$$Y_1(f) = 0, \ldots, Y_q(f) = 0,$$

in $(r+q-1)$ variables which will admit the group $X_1, \ldots, X_{r-1}$; and $X_1, \ldots, X_{r-1}, Y_1, \ldots, Y_q$ will all be unconnected operators.

If the greatest sub-group of $X_1, \ldots, X_{r-1}$ is self-conjugate, we may take this sub-group to be $X_1, \ldots, X_{r-2}$, and thus by quadratures obtain another integral of

$$Y_1(f) = 0, \ldots, Y_q(f) = 0;$$

and hence proceeding find all the integrals by quadratures, provided that each successive maximum sub-group is self-conjugate within the previous one.

§ 88. Suppose we are given the linear differential equation

$$\xi_1 \frac{\partial f}{\partial x_1} + \ldots + \xi_n \frac{\partial f}{\partial x_n} = 0,$$

how far does the method explained help us in obtaining some or all of its integrals?

We know that by a suitable choice of variables the equation may be reduced to the form $\frac{\partial f}{\partial x_1} = 0$; and therefore it will admit any operator whose form in the new variables is

$$\eta_2 \frac{\partial}{\partial x_2} + \ldots + \eta_n \frac{\partial}{\partial x_n},$$

where $\eta_2, \ldots, \eta_n$ are functions of $x_2, \ldots, x_n$ only. Every equation must therefore admit $(n-1)$ reduced unconnected operators; but, since the reduction of a given equation to the form $\frac{\partial f}{\partial x_1} = 0$ would require integration operations of order $(n-1)$, we do not know any general method of obtaining the infinitesimal operators admitted by the given equation.

Lie's method does not therefore apply to any arbitrarily chosen differential equation, but merely to those equations which admit known operators. These operators may be known from the form of the differential equation, or from its geometrical genesis.

When we do know, by any method, the integrals of a given equation, it would be a simple matter to construct infinitesimal transformations which the equation will admit; and then, knowing these infinitesimal transformations, we could solve the equation by Lie's method. Such examples would however merely serve as exercises in applying the method, and could not show its real interest. What is remarkable is that those particular types of differential equations whose solutions

have long been known, and were discovered by various artifices, are equations which do admit obvious infinitesimal transformations, i.e. transformations which would be anticipated without any knowledge of the solution of the equation and merely from its form, or from the geometrical meaning of the equation.

§ 89. Before illustrating the method by a few simple examples it will be necessary to consider how it applies to ordinary equations in two variables.

Consider the equation

$$(1) \qquad y_{n+1} = f(x, y, y_1, \ldots, y_n),$$

where y_r is written for $\dfrac{d^r y}{dx^r}$.

Since $$\frac{dx}{1} = \frac{dy}{y_1} = \frac{dy_1}{y_2} = \ldots = \frac{dy_n}{f(x, y, \ldots, y_n)},$$

we see that the solution of (1) will be obtained only when we have obtained all the invariants of

$$\frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \ldots + f \frac{\partial}{\partial y_n},$$

$x, y, y_1, \ldots, y_n$ being regarded as unconnected variables.

If the equation (1) admits

$$x' = x + t\xi(x, y), \quad y' = y + t\eta(x, y),$$

then we have shown how to extend this point transformation to any required order; and therefore corresponding to any *known* infinitesimal transformation admitted by

$$y_{n+1} = f(x, y, y_1, \ldots, y_n)$$

we shall have a known infinitesimal transformation admitted by

$$\frac{\partial u}{\partial x} + y_1 \frac{\partial u}{\partial y} + \ldots + f \frac{\partial u}{\partial y_n} = 0,$$

and we can therefore reduce the order of the integration operations necessary for the solution of (1).

§ 90. We shall now give one or two simple examples of the application of Lie's method.

Example. Consider the linear equation

$$y_1 + yf(x) = \phi(x),$$

where y_1 is written for $\dfrac{dy}{dx}$.

Let any integral of this equation be $y = \xi$, where ξ is a function of x, and let $y = \xi^0$ be any integral of $y_1 + yf(x) = 0$, then $y = \xi + c\xi^0$, where c is an arbitrary constant, is also an integral; we express this in Lie's notation by saying that the given equation admits the infinitesimal transformation

$$y' = y + t\xi^0, \quad x' = x.$$

The partial differential equation

$$\frac{\partial u}{\partial x} + (\phi(x) - yf(x)) \frac{\partial u}{\partial x} = 0$$

therefore admits the operator $\xi^0 \frac{\partial}{\partial y}$; and, if u is any invariant of $\frac{\partial}{\partial x} + (\phi(x) - yf(x)) \frac{\partial}{\partial y}$, then $\xi_0 \frac{\partial u}{\partial y}$ will also be an invariant, and will therefore be a function of u.

We can then find some invariant v, such that

$$\frac{\partial v}{\partial x} + (\phi(x) - yf(x)) \frac{\partial v}{\partial y} = 0, \quad \xi_0 \frac{\partial v}{\partial y} = 1,$$

and such therefore that $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are known in terms of x and y. We can therefore find v by mere quadratures, and thus deduce the complete primitive from our knowledge of two particular integrals, viz. one of the equation

$$\frac{dy}{dx} + yf(x) = \phi(x),$$

and one of the equation

$$\frac{dy}{dx} + yf(x) = 0.$$

Example. The equation

$$y_1 = f\left(\frac{y}{x}\right)$$

obviously admits the transformation

$$x' = ax, \quad y' = ay,$$

where a is a variable parameter, and therefore

$$\frac{\partial u}{\partial x} + f\left(\frac{y}{x}\right) \frac{\partial u}{\partial y} = 0$$

admits
$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},$$
so that the homogeneous equation of the first order can be solved by quadrature.

Example. Curves whose equations are given in the form of a relation between r and p, where r is the distance of a point on the curve from the origin, and p the perpendicular from the origin to the tangent at the point, can always be solved; that is, we can obtain the Cartesian equation of these curves. These equations are of the form
$$y-xy_1=\sqrt{1+y_1^2}\,f(x^2+y^2),$$
and, from their geometrical meaning, must be unaltered by rotation of the axes of coordinates; that is, they admit the operator $y\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial y}$ and can therefore be solved by quadratures.

§ 91. Euler has shown how to integrate the equation
$$y_1=\frac{b+ex+gy+hxy+ky^2}{a+cx+dy+hx^2+kxy};$$
we shall show how this would be solved by Lie's method.

Writing down the equation
$$(a+cx+dy+hx^2+kxy)\frac{\partial u}{\partial x}+(b+ex+gy+hxy+ky^2)\frac{\partial u}{\partial y}=0,$$
we are to find some infinitesimal transformation which it will admit.

It is obvious that any *projective* transformation must transform this equation into another of the same form, though not necessarily with the same constant coefficients $a, b, c, d, e, f, g, h, k$; we therefore seek that particular *projective* transformation (if such exists) which the equation may admit.

It is now necessary to state a general theorem (the proof will be given later) which will help us in finding the forms of the infinitesimal transformations which a given complete equation system may admit.

Suppose that $Y_1(f)=0, \ldots, Y_q(f)=0$ is a complete equation system of order q and that
$$Y_k=\eta_{k1}\frac{\partial}{\partial x_1}+\ldots+\eta_{kn}\frac{\partial}{\partial x_n},$$

then not all q-rowed determinants of the matrix

$$\left\| \begin{matrix} \eta_{11}, & \cdot & \cdot & \cdot & \eta_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \eta_{q1}, & \cdot & \cdot & \cdot & \eta_{qn} \end{matrix} \right\|$$

can vanish identically.

A point $x_1, \ldots, x_n$ such that, when we substitute its coordinates in the matrix, not all q-rowed determinants of the matrix vanish, is said to be a point of *general* position; a point such that all $(h+1)$-rowed determinants, but not all h-rowed determinants vanish, is said to be a point of *special* position of order h; h may have any value from 1 to q, but if h is equal to q the special point becomes a general point. The theorem, assumed for the present, is that by any transformation, which the given equation system can admit, a point of general position must be transformed into a point of general position; and a point of special position into a point of special position of the same order.

In the example we are considering the points of special position are those points which satisfy the two equations

$$a + cx + dy + hx^2 + kxy = 0, \quad b + ex + gy + hxy + ky^2 = 0.$$

We see that in general there are three points not at infinity, and one point at infinity, common to these two conics; by a linear transformation of coordinates we may take these points to be the points whose coordinates are respectively

$$(0, 0), \quad (0, 1), \quad (1, 0),$$

and in this system of coordinates the equation whose solution is required is

$$(1) \quad \left(a_1(x - x^2) - a_2 xy\right)\frac{\partial u}{\partial x} + \left(a_2(y - y^2) - a_1 xy\right)\frac{\partial u}{\partial y} = 0.$$

Since we are now seeking a projective transformation which the equation will admit, it must be one which will not alter the points

$$(0, 0), \quad (0, 1), \quad (1, 0),$$

and it will therefore be of the form

$$\left(\alpha_1(x - x^2) - \alpha_2 xy\right)\frac{\partial}{\partial x} + \left(\alpha_2(y - y^2) - \alpha_1 xy\right)\frac{\partial}{\partial y},$$

where α_1, α_2 are undetermined constants.

We now easily see that the equation (1) admits

$$(x-x^2)\frac{\partial}{\partial x}-xy\frac{\partial}{\partial y} \quad \text{and} \quad (y-y^2)\frac{\partial}{\partial y}-xy\frac{\partial}{\partial y}.$$

These two operators are not *reduced* unconnected operators, but the knowledge of either is sufficient to reduce the solution of (1) to quadratures.

As our object is to illustrate the uniformity of Lie's method as contrasted with the earlier and more special methods, and not actually to obtain the integrals of differential equations, we shall not carry out the operations necessary to obtain the explicit solution of the equation. It may often be found that the special methods with which we are familiar will obtain the solution of known equations more rapidly than we can obtain them by the more general method of Lie.

§ 92. As an example of Lie's method of depressing equations, take the known result that a differential equation can be depressed when one of the variables is absent. Since, if x does not appear in it, the equation must admit $\frac{\partial}{\partial x}$, and if y does not appear it must admit $\frac{\partial}{\partial y}$, we see that the integration operations necessary for the solution are lowered by unity. So if neither x nor y occur explicitly the order may be depressed by two, for the equation will now admit $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Again, any homogeneous equation can be depressed since it admits

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\dots.$$

Thus, if we take

$$(1)\quad (a_1x+b_1y+c_1z)\frac{\partial u}{\partial x}+(a_2x+b_2y+c_2z)\frac{\partial u}{\partial y} + (a_3x+b_3y+c_3z)\frac{\partial u}{\partial z}=0,$$

since it admits $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$, we must find the common integral of (1) and

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0;$$

eliminating $\frac{\partial u}{\partial z}$ this common integral must satisfy the equation

$$z(a_1x+b_1y+c_1z)\frac{\partial u}{\partial x}+z(a_2x+b_2y+c_2z)\frac{\partial u}{\partial y}$$
$$=(a_3x+b_3y+c_3z)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right).$$

In this equation z occurs only as a parameter, and therefore taking $x'=xz$, $y'=yz$, the equations become

$$(a_1x'+b_1y'+c_1)\frac{\partial u}{\partial x'}+(a_2x'+b_2y'+c_2)\frac{\partial u}{\partial y'}$$
$$=(a_3x'+b_3y'+c_3)\left(x'\frac{\partial u}{\partial x'}+y'\frac{\partial u}{\partial y'}\right).$$

We have proved that the integral of this can be obtained by a quadrature; and therefore u must be of the form

$$F(x,y,z)+\phi(z),$$

where F is a known function and $\phi(z)$ an unknown function.

Since u is annihilated by $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$ the unknown function $\phi(z)$ can also be obtained by quadrature.

Having thus obtained the common integral of the equations, we introduce it as a new variable; it then enters the equation (1) merely as a parameter, in which form it also enters the operator $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$, when this latter is expressed in the new variables.

We thus have an equation in two variables admitting an operator, and can therefore find by a mere quadrature the other integral.

CHAPTER VIII

INVARIANT THEORY OF GROUPS

§ 93. We have already defined transitive groups (§ 44), but it is now convenient to give a second definition of such groups, and to show that the two definitions are consistent.

The group

$$(1) \quad x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

is said to be *transitive* if amongst its operations one can be found which transforms any arbitrarily assigned point into some other point, also arbitrarily assigned.

The group will therefore be transitive if, and only if, the equations (1) can be thrown into such a form, that some n of the parameters $a_1, \ldots, a_r$ can be expressed in terms of $x_1, \ldots, x_n$, $x_1', \ldots, x_n'$ and the remaining parameters. The group cannot then be transitive unless $r \geqq n$. The group will be transitive unless all n-rowed determinants vanish identically in the matrix

$$\left\| \begin{matrix} \dfrac{\partial x_1'}{\partial a_1}, & \cdot & \cdot & \cdot & \dfrac{\partial x_1'}{\partial a_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial x_n'}{\partial a_1}, & \cdot & \cdot & \cdot & \dfrac{\partial x_n'}{\partial a_r} \end{matrix} \right\|.$$

If we recall the rule for forming the infinitesimal operators we shall see that the group is transitive unless every n of those operators are connected; and we thus see that the two definitions are consistent.

The group is transitive therefore if, and only if, it contains n unconnected operators. If $r = n$ the group, if transitive at all, is *simply transitive*; and in this case there are only a discrete number of operations which transform an arbitrarily assigned point into another arbitrarily assigned point.

The mere fact that $r \geqq n$ is not enough to secure the

transitivity of the group; thus we saw that r was equal to n for the group of rotations about the origin, viz.

$$y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z},\quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x},$$

but the group is not transitive, for these operators are connected.

An intransitive group cannot therefore have n unconnected operators. Let such a group have q unconnected operators; we shall now prove that these form a complete system.

Let $X_1, \ldots, X_q$ be any q unconnected operators of the group, and let the other operators be $X_{q+1}, \ldots, X_r$ then

$$X_{q+j}=\phi_{q+j,1}X_1+\ldots+\phi_{q+j,q}X_q,\qquad (j=1,\ldots,r-q),$$

where $\phi_{q+j,k}, \ldots$ are known functions of $x_1, \ldots, x_n$.

We have

$$(X_i, X_k)=\sum^{s=r} c_{iks}X_s=\sum^{s=q}\left(c_{iks}+\sum^{j=r-q} c_{i,k,q+j}\,\phi_{q+j,s}\right)X_s,$$

where i and k may have any values from 1 to q, and therefore $X_1, \ldots, X_q$ form a complete system.

If a function is annihilated by these q operators $X_1, \ldots, X_q$, it must also be annihilated by $X_{q+1}, \ldots X_r$; and therefore on considering the canonical form of the group we see that such a function is unaltered by any transformation of the group. We have proved that there are $(n-q)$ functions annihilated by $X_1, \ldots, X_q$, and we therefore conclude that an intransitive group has $(n-q)$ unconnected invariants.

§ 94. To express this result geometrically we look on $x_1, \ldots, x_n$ as the coordinates of a point in n-way space, then

$$f_1(x_1, \ldots, x_n)=a_1, \ldots, f_{n-q}(x_1, \ldots, x_n)=a_{n-q}$$

will be a q-way locus in this space, and the coordinates of this locus are the constants $a_1, \ldots, a_{n-q}$. We keep the form of the functions $f_1, \ldots, f_{n-q}$ fixed, but vary the constants, and thus have these q-way loci (or q-folds) passing through every point of space. If we take $f_1, \ldots, f_{n-q}$ to be the invariants of the intransitive groups, then by the operations of the group a point lying on one of these loci is moved to some other point on that locus; we say therefore that this decomposition of space, into ∞^{n-q} q-folds, is invariant under all the operations of the group. Thus for the group of rotations about the

origin, space is decomposed into a simple infinity of spheres, whose centre is the origin, and a point lying on any one of these spheres can only be transformed to some other point on the same sphere.

§ 95. Only intransitive groups can strictly be said to have invariants, and the problem of finding these invariants is equivalent to that of finding the integrals of the complete equation system formed by their unconnected operators; yet we shall see that in several ways the idea of invariants can be extended to transitive groups also. Two points of space, $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, which are transformed to two other points by the same transformation scheme, are said to be transformed *cogrediently*; thus if

$$x_i' = f_i(x_1, \ldots, x_n,\ a_1, \ldots, a_r),$$
$$y_i' = f_i(y_1, \ldots, y_n,\ a_1, \ldots, a_r),$$

we should say that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ were transformed cogrediently.

No function of the coordinates of a point is invariant for the operations of a transitive group, yet there may be functions of the coordinates of a pair of points, which are invariant when the points are transformed cogrediently by the operations of a transitive group; thus the transitive group

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$$

has the three invariants $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$, where x_1, y_1, z_1 and x_2, y_2, z_2 are two points cogrediently transformed by this translation group.

We could say in this case that we have extended the point group

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$$

into the point-pair group

$$\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\quad \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2},\quad \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2};$$

and this extended group is intransitive, and has the three unconnected invariants $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$.

Similarly the group of movements of a rigid body, viz.

$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\quad y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

is transitive and has no invariant; yet when extended so as to give the point-pair group

$$\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2},\ldots,\; y_1\frac{\partial}{\partial z_1}-z_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial z_2}-z_2\frac{\partial}{\partial y_2},\ldots$$

this group is intransitive, and has the invariant

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2.$$

This expression is therefore an invariant of the coordinates of a point-pair, when cogrediently transformed by the operations of the transitive group of movements of a rigid body.

The reason why this extended group of six operators in six variables has an invariant is that the operators are *connected*, as we prove by considering the determinant

$$\begin{vmatrix} 1, & 0, & 0, & 1, & 0, & 0 \\ 0, & 1, & 0, & 0, & 1, & 0 \\ 0, & 0, & 1, & 0, & 0, & 1 \\ 0, & -z_1, & y_1, & 0, & -z_2, & y_2 \\ z_1, & 0, & -x_1, & z_2, & 0, & -x_2 \\ -y_1, & x_1, & 0, & -y_2, & x_2, & 0 \end{vmatrix},$$

and subtracting the first column from the fourth, the second from the fifth, and the third from the sixth, when it is seen to be zero.

Since five of the operators are unconnected there is no other unconnected invariant of a point-pair for the operations of the group of movements.

If we were to extend this group so as to apply to triplets of points we should not get any really new invariants; it is only when the operators are taken so as to apply to point-pairs that the six operators are connected; in the case of point-triplets we should have six unconnected operators in nine variables; and therefore only three invariants, viz. the expressions for the mutual distances of these points.

§ 96. The operators of the linear group of the plane, viz.

$$x'=l_1x+m_1y,\quad y'=l_2x+m_2y,$$

are
$$x\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial x},\quad y\frac{\partial}{\partial y};$$

and as two of these are unconnected the group has no invariant.

If, however, $a_0x^p+pa_1x^{p-1}y+\dots$

is any binary quantic, the quantic becomes, on applying the transformations of the group,

$$a_0'x'^p+pa_1'x'^{p-1}y'+\dots;$$

and we often speak of those functions of the coefficients $a_0, a_1, \dots$, which are such that

$$f(a_0, a_1, \dots)=f(a_0', a_1', \dots),$$

as invariants of the linear group.

These functions are however invariants, not of the linear group

$$x'=l_1x+m_1y,\quad y'=l_2x+m_2y,$$

but of the group

$$a_0'=a_0l_1^p+\dots,\quad a_1'=a_0l_1^{p-1}m_1+\dots,\quad a_2'=a_0l_1^{p-2}m_1^2+\dots,$$

of which the linear operators are A_1, A_2, A_3, A_4, where

$$A_1=pa_0\frac{\partial}{\partial a_0}+(p-1)\,a_1\frac{\partial}{\partial a_1}+(p-2)\,a_2\frac{\partial}{\partial a_2}+\dots+a_{p-1}\frac{\partial}{\partial a_{p-1}},$$

$$A_2=pa_1\frac{\partial}{\partial a_0}+(p-1)\,a_2\frac{\partial}{\partial a_1}+(p-2)\,a_3\frac{\partial}{\partial a_2}+\dots+a_p\frac{\partial}{\partial a_{p-1}},$$

$$A_3=\ a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\dots+pa_{p-1}\frac{\partial}{\partial a_p},$$

$$A_4=\ a_1\frac{\partial}{\partial a_1}+2a_2\frac{\partial}{\partial a_2}+3a_3\frac{\partial}{\partial a_3}+\dots+pa_p\frac{\partial}{\partial a_p}.$$

If we denote the operators $x\frac{\partial}{\partial x}$ by X_1, $x\frac{\partial}{\partial y}$ by X_2, $y\frac{\partial}{\partial x}$ by X_3, and $y\frac{\partial}{\partial y}$ by X_4, we see that

$$X_1-A_1,\ X_2-A_2,\ X_3-A_3,\ X_4-A_4$$

are four operators, each of which annihilates the quantic

$$a_0x^p+pa_1x^{p-1}y+\dots;$$

and that there is no operator of the form

$$a_0\frac{\partial}{\partial a_0}+a_1\frac{\partial}{\partial a_1}+a_2\frac{\partial}{\partial a_2}+\dots$$

(where $a_0, a_1, \ldots$ are functions of the coefficients $a_0, a_1, \ldots$ only) which will annihilate this quantic.

We must now express the invariant theory of binary quantics in such a form as to suggest the extension to general group theory.

First we verify the group property of X_1, X_2, X_3, X_4 by noticing that

$$(X_1, X_2) = X_2, \quad (X_1, X_3) = -X_3, \quad (X_1, X_4) = 0,$$
$$(X_2, X_3) = X_1 - X_4, \quad (X_2, X_4) = X_2, \quad (X_3, X_4) = -X_3.$$

Next we see that the operator

$$(X_i - A_i, X_k - A_k) - \sum^{h=4} c_{ikh}(X_h - A_h), \qquad \binom{i = 1, \ldots, 4}{k = 1, \ldots, 4}$$

annihilates the quantic, since each operator

$$X_1 - A_1, \; X_2 - A_2, \; X_3 - A_3, \; X_4 - A_4$$

annihilates it.

Since $X_1, \ldots, X_4$ are each commutative with $A_1, \ldots, A_4$ (being operators in different sets of variables), and since by the group property

$$(X_i, X_k) - \sum^{h=4} c_{ikh} X_h \equiv 0,$$

we conclude that

$$(-A_i, -A_k) - \sum^{h=4} c_{ikh}(-A_h)$$

must annihilate the quantic.

Now this is a linear operator, not containing x or y; it can therefore only annihilate the quantic if the coefficients of $\frac{\partial}{\partial a_0}, \ldots, \frac{\partial}{\partial a_p}$ in it are identically zero: we conclude that

$$(-A_i, -A_k) \equiv \sum^{h=4} c_{ikh}(-A_h);$$

that is, the operators $-A_1, -A_2, -A_3, -A_4$ generate a group, and this group has the same structure constants as the group X_1, X_2, X_3, X_4.

§ 97. We shall now take X to denote the linear operator

$$e_1 X_1 + e_2 X_2 + e_3 X_3 + e_4 X_4,$$

and A to denote the linear operator

$$e_1 A_1 + e_2 A_2 + e_3 A_3 + e_4 A_4,$$

where e_1, e_2, e_3, e_4 are parameters unconnected with the coefficients or variables in the binary quantic.

Since $X - A$ annihilates the quantic we have

$$a_0 x^p + pa_1 x^{p-1} y + \ldots = e^{X-A}(a_0 x^p + pa_1 x^{p-1} y + \ldots),$$
$$= e^{-A} e^{X}(a_0 x^p + pa_1 x^{p-1} y + \ldots),$$

any operator X_i being commutative with any operator A_j.

The linear transformation

$$(1) \qquad x' = e^X x, \quad y' = e^Y y$$

gives $\quad e^X(a_0 x^p + pa_1 x^{p-1} y + \ldots) = a_0 x'^p + pa_1 x'^{p-1} y' + \ldots;$

and therefore, since

$$a_0 x^p + pa_1 x^{p-1} y + \ldots = a_0' x'^p + pa_1' x'^{p-1} y' + \ldots,$$

we conclude that

$$e^{-A}(a_0 x'^p + pa_1 x'^{p-1} y' + \ldots) = a_0' x'^p + pa_1' x'^{p-1} y' + \ldots.$$

Equating coefficients of like powers of the variables on each side, we see that

$$(2) \qquad a_i' = e^{-A} a_i,$$

and so generally $\phi(a_0', a_1', \ldots) = e^{-A}\phi(a_0, a_1, \ldots)$.

It now follows from (1) and (2) that if

$$f(x, y, a_0, a_1, \ldots, a_p)$$

is any function whatever of $x, y, a_0, a_1, \ldots$

$$f(x', y', a_0', a_1', \ldots, a_p') = e^{X-A} f(x, y, a_0, a_1, \ldots, a_p).$$

§ 98. Covariants and Invariants, as defined in the Algebra of Quantics, are therefore merely the functions annihilated by

$$X_1 - A_1, \ldots, X_4 - A_4,$$

four operators which are unconnected, and which generate a finite continuous group.

If we are given a group $X_1, \ldots, X_r$ and want to find the invariant theory which will bear the same relation to this group as the invariant theory of the Algebra of Quantics bears to the linear group, we must find some function

$$\phi(x_1, \ldots, x_n, c_1, \ldots, c_m),$$

where $c_1, \ldots, c_m$ are constants, such that for any transformation of the group we may have the fundamental identity

$$\phi(x_1', \ldots, x_n', c_1', \ldots, c_m') = \phi(x_1, \ldots, x_n, c_1, \ldots, c_m),$$

$c'_1, \ldots, c'_m$ being constants, which are functions of $c_1, \ldots, c_m$ and the parameters $a_1, \ldots, a_r$ of the given group.

Following the analogy of the procedure in the theory of binary quantics we should only take such a function as satisfied no equation of the form

$$\gamma_1 \frac{\partial \phi}{\partial c_1} + \ldots + \gamma_m \frac{\partial \phi}{\partial c_m} = 0,$$

where $\gamma_1, \ldots, \gamma_m$ are functions of $c_1, \ldots, c_m$ only.

If the function found did satisfy such an equation we could (since in it the parameters would not occur effectively) replace it by a function containing fewer parameters.

Suppose now that we have found a function, with m effective parameters, satisfying the fundamental identity

$$\phi(x'_1, \ldots, x'_n, c'_1, \ldots, c'_m) = \phi(x_1, \ldots, x_n, c_1, \ldots, c_m).$$

Applying the identical transformation

$$x'_i = x_i, \qquad (i = 1, \ldots, n),$$

we have for it

$$\phi(x_1, \ldots, x_n, c'_1, \ldots, c'_m) = \phi(x_1, \ldots, x_n, c_1, \ldots, c_m);$$

and therefore, since $x_1, \ldots, x_n$ are unconnected,

$$c'_k = c_k, \qquad (k = 1, \ldots, m).$$

We next apply the infinitesimal transformation

$$x'_i = x_i + t\xi_{ki}(x_1, \ldots, x_n), \qquad \left(\begin{matrix} i = 1, \ldots, n \\ k = 1, \ldots, r \end{matrix}\right),$$

and we must have, since c'_k is a function of $c_1, \ldots, c_m$ and differs infinitesimally from c_k,

$$c'_k = c_k + t\gamma_{hk}(c_1, \ldots, c_m), \qquad \left(\begin{matrix} h = 1, \ldots, r \\ k = 1, \ldots, m \end{matrix}\right),$$

where $\gamma_{hk}, \ldots$ are functions of $c_1, \ldots, c_m$.

If then we denote by C_k the operator

$$\gamma_{k1} \frac{\partial}{\partial c_1} + \ldots + \gamma_{km} \frac{\partial}{\partial c_m},$$

we see that

$$X_1 + C_1, \ldots, X_r + C_r$$

will each annihilate $\phi(x_1, \ldots, x_n, c_1, \ldots, c_m)$.

Proceeding as in the theory of binary quantics the operator

$$(X_i+C_i,\ X_k+C_k)-\sum^{h=r} c_{ikh}(X_h+C_h)$$

is seen to annihilate this function. Since no operator in $c_1, \ldots, c_m$ only can do this, and since $X_1, \ldots, X_r$ are commutative with $C_1, \ldots, C_r$, we conclude that

$$(C_i,\ C_k)=\sum^{h=r} c_{ikh}C_h;$$

and therefore $C_1, \ldots, C_r$ generate a group with the same structure constants as the group $X_1, \ldots, X_r$.

We do not, however, know that the operators $C_1, \ldots, C_r$ will be *independent*; and therefore the group which they generate may be of an order less than r.

Since

$$X_1+C_1, \ldots, X_r+C_r$$

generate a group, all of whose operators annihilate

$$\phi(x_1, \ldots, x_n,\ c_1, \ldots, c_m),$$

this group must be intransitive.

§ 99. When we are given the group $X_1, \ldots, X_r$ we can construct many functions of $x_1, \ldots, x_n$ and a set of parameters $c_1, \ldots, c_m$, which will have the fundamental property of possessing an invariant theory; it will be sufficient to show how one such function may be obtained.

Let $A_1, \ldots, A_r$, operators in the variables $a_1, \ldots, a_r$, be the parameter group of $X_1, \ldots, X_r$; and let $B_1, \ldots, B_r$ be the same parameter group, but written in the variables $b_1, \ldots, b_r$ instead of $a_1, \ldots, a_r$; then

$$(1)\qquad X_1+A_1+B_1, \ldots, X_r+A_r+B_r$$

is a group with r unconnected operators. This group must therefore have $(n+r)$ unconnected invariants, for it is a group of order r in $(n+2r)$ variables.

If some one of these invariants does not involve $x_1, \ldots, x_n$ it must be an invariant of the operators

$$A_1+B_1, \ldots, A_r+B_r;$$

and as there are r invariants of this group, we see that there must be n invariants of (1) which will be *unconnected* functions of $x_1, \ldots, x_n$, but may also involve the parameters

$a_1, \ldots, a_r,\ b_1, \ldots, b_r$ in addition to the variables $x_1, \ldots, x_n$; and some one at least of these invariants must do so; else would $X_1, \ldots, X_r$ annihilate each of the variables $x_1, \ldots, x_n$ which is of course impossible.

We thus see that for any group there must always be a function with the fundamental property

$$(2) \quad \phi(x_1', \ldots, x_n',\ c_1', \ldots, c_m') = \phi(x_1, \ldots, x_n,\ c_1, \ldots, c_m);$$

and therefore an invariant theory for each group.

The reason why we take the operators

$$X_1 + A_1 + B_1, \ldots, X_r + A_r + B_r$$

rather than the operators

$$X_1 + A_1, \ldots, X_r + A_r,$$

is that for the latter set of operators there can be no invariant theory; since, $A_1, \ldots, A_r$ being a transitive group, there are no functions of $a_1, \ldots, a_r$ annihilated by these operators.

We now take X and C to denote the respective operators

$$e_1 X_1 + \ldots + e_r X_r \quad \text{and} \quad e_1 C_1 + \ldots + e_r C_r;$$

and, as in the corresponding theory for binary quantics, we have, since $X + C$ annihilates $\phi(x_1, \ldots, x_n,\ c_1, \ldots, c_m)$,

$$\begin{aligned} \phi(x_1, \ldots, x_n,\ c_1, \ldots, c_m) &= e^{X+C}\phi(x_1, \ldots, x_n,\ c_1, \ldots, c_m), \\ &= e^C e^X \phi(x_1, \ldots, x_n,\ c_1, \ldots, c_m); \end{aligned}$$

and therefore

$$\phi(x_1', \ldots, x_n',\ c_1', \ldots, c_m') = e^C \phi(x_1', \ldots, x_n',\ c_1, \ldots, c_m).$$

Since the parameters $c_1, \ldots, c_m$ enter the fundamental function ϕ effectively, we now have

$$c_i' = e^C c_i, \qquad (i = 1, \ldots, m);$$

and more generally, if $f(x_1, \ldots, x_n,\ c_1, \ldots, c_m)$ is any function whatever, we must have

$$f(x_1', \ldots, x_n',\ c_1', \ldots, c_m') = e^{X+C} f(x_1, \ldots, x_n,\ c_1, \ldots, c_m).$$

The covariants are therefore those functions of $x_1, \ldots, x_n$, $c_1, \ldots, c_m$ which are annihilated by

$$X_1 + C_1, \ldots, X_r + C_r;$$

and the invariants are those functions of $c_1, \ldots, c_m$ which are annihilated by

$$C_1, \ldots, C_r;$$

and therefore for every group we have a corresponding invariant theory.

§ 100. For a given group $X_1, \ldots, X_r$ we may be able to obtain a fundamental function without having to go through the process of finding $C_1, \ldots, C_r$, and then finding the invariants of $X_1 + C_1, \ldots, X_r + C_r$.

Thus if we take the group of order ten $X_1, \ldots, X_{10}$, where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

$$X_5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad X_6 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

$$X_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

$$X_8 = (y^2+z^2-x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z},$$

$$X_9 = (z^2+x^2-y^2)\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial z},$$

$$X_{10} = (x^2+y^2-z^2)\frac{\partial}{\partial z} - 2yz\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z},$$

a group which transforms minimum curves into minimum curves, we see that by any operation of this group the function

$$(1) \qquad \frac{a_1(x^2+y^2+z^2) + 2g_1x + 2f_1y + 2h_1z + d_1}{a_2(x^2+y^2+z^2) + 2g_2x + 2f_2y + 2h_2z + d_2}$$

is transformed into a function of like form, but with a different set of constants.

The function (1) being fundamental, the group in the parameters is $C_1, \ldots, C_{10}$ where

$$C_1 = -a_1\frac{\partial}{\partial g_1} - 2g_1\frac{\partial}{\partial d_1} - a_2\frac{\partial}{\partial g_2} - 2g_2\frac{\partial}{\partial d_2},$$

$$C_2 = -a_1\frac{\partial}{\partial f_1} - 2f_1\frac{\partial}{\partial d_1} - a_2\frac{\partial}{\partial f_2} - 2f_2\frac{\partial}{\partial d_2},$$

$$C_3 = -a_1\frac{\partial}{\partial h_1} - 2h_1\frac{\partial}{\partial d_1} - a_2\frac{\partial}{\partial h_2} - 2h_2\frac{\partial}{\partial d_2},$$

$$C_4 = -h_1\frac{\partial}{\partial f_1} + f_1\frac{\partial}{\partial h_1} - h_2\frac{\partial}{\partial f_2} + f_2\frac{\partial}{\partial h_2},$$

$$C_5 = -g_1\frac{\partial}{\partial h_1} + h_1\frac{\partial}{\partial g_1} - g_2\frac{\partial}{\partial h_2} + h_2\frac{\partial}{\partial g_2},$$

$$C_6 = -f_1\frac{\partial}{\partial g_1} + g_1\frac{\partial}{\partial f_1} - f_2\frac{\partial}{\partial g_2} + g_2\frac{\partial}{\partial f_2},$$

$$C_7 = g_1\frac{\partial}{\partial g_1} + f_1\frac{\partial}{\partial f_1} + h_1\frac{\partial}{\partial h_1} + g_2\frac{\partial}{\partial g_2} + f_2\frac{\partial}{\partial f_2} + h_2\frac{\partial}{\partial h_2} + 2d_1\frac{\partial}{\partial d_1} + 2d_2\frac{\partial}{\partial d_2},$$

$$C_8 = -2g_1\frac{\partial}{\partial a_1} - d_1\frac{\partial}{\partial g_1} - 2g_2\frac{\partial}{\partial a_2} - d_2\frac{\partial}{\partial g_2},$$

$$C_9 = -2f_1\frac{\partial}{\partial a_1} - d_1\frac{\partial}{\partial f_1} - 2f_2\frac{\partial}{\partial a_2} - d_2\frac{\partial}{\partial f_2},$$

$$C_{10} = -2h_1\frac{\partial}{\partial a_1} - d_1\frac{\partial}{\partial h_1} - 2h_2\frac{\partial}{\partial a_2} - d_2\frac{\partial}{\partial h_2}.$$

It may be verified that this group has the same structure as $X_1, \ldots, X_{10}$.

This group, though of the tenth order and in ten variables, is intransitive, and has the absolute invariant

$$\frac{(2g_1g_2 + 2f_1f_2 + 2h_1h_2 - d_1a_2 - a_2d_1)^2}{(g_1^2 + f_1^2 + h_1^2 - a_1d_1)(g_2^2 + f_2^2 + h_2^2 - a_2d_2)}.$$

Since the group $X_1, \ldots, X_{10}$ transforms spheres into spheres, and surfaces intersecting at any angle into surfaces intersecting at the same angle, we could have foreseen that the group must have this invariant, for it is a function of the angle at which the two spheres,

$$a_1(x^2+y^2+z^2) + 2g_1x + 2f_1y + 2h_1z + d_1 = 0,$$
$$a_2(x^2+y^2+z^2) + 2g_2x + 2f_2y + 2h_2z + d_2 = 0,$$

intersect.

§ 101. We know that only intransitive groups can properly be said to have invariant functions, but groups, whether transitive or intransitive, may have *invariant equations.*

Before we consider the theory of the invariant equations admitting a given group, we must prove the theorem quoted in § 91 as to the transformations which a complete equation system can admit.

Let $Y_1, \ldots, Y_q$ be the operators of a complete system where

$$Y_k = \eta_{k1}\frac{\partial}{\partial x_1} + \ldots + \eta_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, q),$$

and let $Y'_i, \ldots, Y'_q$ be the corresponding operators obtained by replacing x_i by x'_i in $Y_1, \ldots, Y_q$, where

$$x'_i = \phi_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

is any transformation scheme.

We know that the equation system admits this transformation if, and only if,

$$(1) \quad Y'_k = \rho_{k1} Y_1 + \ldots + \rho_{kq} Y_q, \qquad (k = 1, \ldots, q),$$

where $\rho_{ki}, \ldots$ are functions of $x_1, \ldots, x_n$ such that the determinant

$$\begin{vmatrix} \rho_{11}, & \cdot & \cdot & \cdot & \rho_{1q} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{q1}, & \cdot & \cdot & \cdot & \rho_{qq} \end{vmatrix}$$

does not vanish.

Let η^0_{ik} denote the result of substituting $x^0_1, \ldots, x^0_n$ for $x_1, \ldots, x_n$ respectively in η_{ik}; and let the operator

$$\eta^0_{k1} \frac{\partial}{\partial x_1} + \ldots + \eta^0_{kn} \frac{\partial}{\partial x_n}$$

be denoted by Y^0_k.

If $$x'^0_i = \phi_i(x^0_1, \ldots, x^0_n), \qquad (i = 1, \ldots, n),$$

we shall denote by Y'^0_k the operator

$$\eta'^0_{k1} \frac{\partial}{\partial x'_1} + \ldots + \eta'^0_{kn} \frac{\partial}{\partial x'_n}.$$

Suppose now that $x^0_1, \ldots, x^0_n$ is a point of order h, so that not all h-rowed determinants vanish in the matrix

$$\left\Vert \begin{matrix} \eta^0_{11}, & \cdot & \cdot & \cdot & \eta^0_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \eta^0_{h1}, & \cdot & \cdot & \cdot & \eta^0_{hn} \end{matrix} \right\Vert,$$

then exactly h of the operators $Y^0_1, \ldots, Y^0_q$ are unconnected, viz. $Y^0_1, \ldots, Y^0_h$; what we have to prove is in effect that h of the operators $Y'^0_1, \ldots, Y'^0_q$ will be unconnected.

We have

$$Y^0_{h+j} = \sigma^0_{h+j,1} Y^0_1 + \ldots + \sigma^0_{h+j,h} Y^0_h, \qquad (j = 1, \ldots, q-h),$$

where the functions $\sigma^0_{h+j,k}, \ldots$ are functions of $x^0_1, \ldots, x^0_n$ such

that none of them are infinite; we also suppose that in the neighbourhood of this point all the functions $\eta_{ik}, \ldots$ are regular; that is, we assume that $\eta_{ik} = \eta^0_{ik} +$ a series of powers and products of $(x_1 - x^0_1), \ldots, (x_n - x^0_n)$, and that in this neighbourhood the functions $\rho_{ki}, \ldots$ are regular and their determinant does not vanish; and finally we assume that the transformation

$$x'_i = \phi_i(x_1, \ldots, x_n)$$

is regular in this neighbourhood, so that $\eta'_{ik}, \ldots$ are also regular in the neighbourhood of $x'^0_1, \ldots, x'^0_n$.

We now have

$$Y_k = Y^0_k + \xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, q),$$

where the functions $\xi_{kj}, \ldots$ vanish for $x_i = x_i^0$; and therefore

$$Y'_k = \sum^{j=q} \rho_{kj} Y_j = \sum^{j=h} (\rho^0_{kj} + \sum^{t=h-q} \rho^0_{k.\,h+t}\,\sigma^0_{h+t.\,j})\, Y^0_j$$
$$+ \zeta_{k1}\frac{\partial}{\partial x_1} + \ldots + \zeta_{kn}\frac{\partial}{\partial x_n},$$

where the functions $\zeta_{kj}, \ldots$ vanish for $x_i = x^0_i$.

We can therefore, if we take any $(h+1)$ of these operators $Y_1, \ldots, Y_q$, say $Y'_1, \ldots, Y'_{h+1}$, find functions $\theta^0_1, \ldots, \theta^0_{h+1}$ of $x^0_1, \ldots, x^0_n$ such that

$$\theta^0_1 Y'_1 + \ldots + \theta^0_{h+1} Y'_{h+1} \equiv \xi_1\frac{\partial}{\partial x_1} + \ldots + \xi_n\frac{\partial}{\partial x_n},$$

where $\xi_1, \ldots, \xi_n$ vanish for $x_i = x^0_i$; and therefore

$$\theta^0_1 \eta'_{1j} + \ldots + \theta^0_{h+1}\eta'_{h+1,\,j}, \qquad (j = 1, \ldots, n)$$

is a function of $x'_1, \ldots, x'_n$, $x^0_1, \ldots, x^0_n$ which vanishes when $x_i = x^0_i$; and therefore, since $x'_i = x'^0_i$, if $x_i = x^0_i$, it vanishes for $x'_i = x'^0_i$.

We have thus proved that any $(h+1)$ of the operators $Y'^0_1, \ldots, Y'^0_1$, are connected, for we have proved that all $(h+1)$-rowed determinants vanish in the matrix

$$\left\| \begin{matrix} \eta'^0_{11}, & . & . & . & \eta'^0_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \eta'^0_{q1}, & . & . & . & \eta'^0_{qn} \end{matrix} \right\|.$$

Suppose now that only $(h-s)$ of the operators $Y'^0_1, \ldots, Y'^0_q$ are

unconnected; then just as, from the fact that exactly h of the operators $Y_1^0, \ldots, Y_q^0$ were unconnected, we proved that any $(h+1)$ of the operators $Y_1'^0, \ldots, Y_q'^0$ were connected, so we could now prove that $(h-s+1)$ of the operators $Y_1^0, \ldots, Y_q^0$ are connected, and therefore s cannot exceed zero, so that exactly h of the operators $Y_1'^0, \ldots, Y_q'^0$ are connected.

We have thus proved the theorem that, *by any transformation which a complete system admits, a point of any assigned order is transformed to a point of the same order, provided that the transformation is regular in the neighbourhood of the point.*

§ 102. We now take $X_1, \ldots, X_r$ to be the operators of a group where

$$X_k = \xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, r),$$

and we say, as in the theory of complete systems, that a point is of order h, if when we substitute its coordinates in the matrix

$$\left\| \begin{matrix} \xi_{11}, & \cdot & \cdot & \cdot & \xi_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_{r1}, & \cdot & \cdot & \cdot & \xi_{rn} \end{matrix} \right\|,$$

all $(h+1)$-rowed determinants, but not all h-rowed determinants of this matrix, vanish.

We shall prove later that for any transformation of the group $x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_n), \qquad (i = 1, \ldots, n)$

we shall have

$$X_j' = e_{j1}X_1 + \ldots + e_{jr}X_r, \qquad (j = 1, \ldots, r),$$

where $e_{jk}, \ldots$ are constants whose determinant does not vanish.

If then $x_1^0, \ldots, x_n^0$ is a point of order h all the functions $\xi_{ik}, \ldots$ are regular in its neighbourhood; and, since now no exceptional case can arise through a want of regularity in any of the coefficients, we see, as in the case of the complete system, that by any transformation of the group a point of order h is transformed to a point of order h.

A point of general position is a point of order q; there are ∞^n of such points, for all $(q+1)$-rowed determinants of the matrix vanish identically, where q is the number of unconnected operators; if the group is transitive $q = n$. As there

may be no values of $x_1, \ldots, x_n$ which make all q-rowed determinants vanish, there may be no special points in connexion with an assigned group; if there are such points, there may be a discrete number of them or there may be an infinity of them; if only a discrete number these points must clearly be fixed points, unaltered by any operation of the given group.

Suppose that ∞^s points will make all $(h+1)$-rowed determinants of the matrix vanish, but not all h-rowed determinants vanish; and let

$$(1) \quad x_{s+m} = \phi_{s+m}(x_1, \ldots, x_s), \qquad (m = 1, \ldots, n-s)$$

be the equations which define these points; the theorem which we have proved asserts that points satisfying these equations will be transformed to other points satisfying the same equations; in other words the equations (1) *admit* the operations of the group $X_1, \ldots, X_r$; that is, these equations are *invariant equations*.

§ 103. Let

$$(1) \quad x_{s+m} = \phi_{s+m}(x_1, \ldots, x_s), \qquad (m = 1, \ldots, n-s)$$

be any equation system admitting a group $X_1, \ldots, X_r$; we shall now define a set of operators closely connected with the system.

If $f(x_1, \ldots, x_n)$ is any function of $x_1, \ldots, x_n$, we shall denote by $\bar{f}$ the function $f(x_1, \ldots, x_s, \phi_{s+1}, \ldots, \phi_n)$ of the variables $x_1, \ldots, x_s$; and by $\overline{X}_1, \ldots, \overline{X}_r$ the r operators

$$\bar{\xi}_{k1}\frac{\partial}{\partial x_1} + \ldots + \bar{\xi}_{ks}\frac{\partial}{\partial x_s}, \qquad (k = 1, \ldots, r);$$

we call $\overline{X_1}, \ldots, \overline{X_r}$ the *contracted* operators of $X_1, \ldots, X_r$ with respect to the equation system (1).

From the definition of the bar

$$\frac{\partial}{\partial x_i}\bar{f} = \overline{\left(\frac{\partial f}{\partial x_i}\right) + \sum^{m=n-s} \left(\frac{\partial f}{\partial x_{s+m}}\right)} \left(\frac{\partial \phi_{s+m}}{\partial x_i}\right), \qquad (i = 1, \ldots, s);$$

and therefore

$$\overline{X_k} \cdot \bar{f} = \overline{(X_k x_1)}\,\overline{\left(\frac{\partial f}{\partial x_i}\right)} + \ldots + \overline{(X_k x_s)}\,\overline{\left(\frac{\partial f}{\partial x_s}\right)} + \overline{\sum^{m=n-s} (X_k \phi_{s+m})}\,\overline{\left(\frac{\partial f}{\partial x_{s+m}}\right)};$$

but we also know that

$$\overline{(X_k f)} = \overline{(X_k x_1)}\,\overline{\left(\frac{\partial f}{\partial x_1}\right)} + \dots$$
$$+ \overline{(X_k x_s)}\,\overline{\left(\frac{\partial f}{\partial x_s}\right)} + \sum^{m=n-s} \overline{(X_k\, x_{s+m})}\,\overline{\left(\frac{\partial f}{\partial x_{s+m}}\right)},$$

so that

$$(2)\quad \overline{(X_k f)} = \overline{X_k}\,.\,\bar{f} + \sum^{m=n-s} \overline{(X_k(x_{s+m} - \phi_{s+m}))}\,\overline{\left(\frac{\partial f}{\partial x_{s+m}}\right)}.$$

Now the equations (1) admit the group, and therefore in particular admit the infinitesimal transformations, so that we must have

$$\overline{(X_k(x_{s+m} - \phi_{s+m}))} \equiv 0;$$

and therefore from (2)

$$\overline{(X_k f)} \equiv \overline{X_k}\,.\,\bar{f}, \qquad (k = 1, \dots, r);$$

that is, the result of first operating with X_k on any function of the variables, and then deducing the corresponding function with the bar, is the same as that of first obtaining the function $\bar{f}$, and then operating with the contracted operator $\overline{X_k}$.

§ 104. We can now prove that $\overline{X_1}, \dots, \overline{X_r}$ *generate a group.* From the second fundamental theorem

$$(X_i, X_j) = \sum^{k=r} c_{ijk}\, X_k,$$

and therefore $\quad X_i\, \xi_{jm} - X_j\, \xi_{im} = \sum^{k=r} c_{ijk}\, \xi_{km};$

consequently we must have

$$\overline{X_i\, \xi_{jm}} - \overline{X_j\, \xi_{im}} = \sum^{k=r} c_{ijk}\, \overline{\xi_{km}},$$

and therefore from what we have just proved

$$\overline{X_i}\,.\,\overline{\xi_{jm}} - \overline{X_j}\,.\,\overline{\xi_{im}} = \sum^{k=r} c_{ijk}\, \overline{\xi_{km}},$$

that is, $\quad (\overline{X_i}, \overline{X_j}) = \sum^{k=r} c_{ijk}\, \overline{X_k}.$

It is not, however, necessarily true that the r contracted operators will be *independent.*

If the equations

$$(1)\quad x_{s+m} = \phi_{s+m}(x_1, \ldots, x_s), \qquad (m = 1, \ldots, n-s)$$

are taken to be the equations which define points of order h with respect to the group, $X_1, \ldots, X_r$, we know that these equations will be invariant under the operations of the group; we shall now prove that h of the operators $\overline{X_1}, \ldots, \overline{X_r}$ are *unconnected.*

From the definition of a special point of order h, exactly h of the operators

$$\overline{\xi_{k1}}\frac{\partial}{\partial x_1} + \ldots + \overline{\xi_{kn}}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, r)$$

are unconnected; and therefore not more than h of the operators $\overline{X_1}, \ldots, \overline{X_r}$ can be unconnected.

Also since the equations (1) admit the group $X_1, \ldots, X_r$

$$\overline{\xi_{k,s+m}} = \overline{\left(\frac{\partial \phi_{s+m}}{\partial x_1}\right)}\overline{\xi_{k1}} + \overline{\left(\frac{\partial \phi_{s+m}}{\partial x_2}\right)}\overline{\xi_{k2}} + \ldots + \overline{\left(\frac{\partial \phi_{s+m}}{\partial x_s}\right)}\overline{\xi_{ks}},$$
$$(k = 1, \ldots, r),$$

and from these equations it follows that not less than h of the operators $\overline{X_1}, \ldots, \overline{X_r}$ can be unconnected; we therefore conclude that exactly h of these operators are unconnected.

§ 105. We are now in a position to determine all the equation systems admitting a given group.

If the system of equations

$$(1)\quad x_{s+m} = \phi_{s+m}(x_1, \ldots, x_s), \qquad (m = 1, \ldots, n-s)$$

is to admit all the transformations, it must in particular admit all the infinitesimal transformations of the group, and therefore we must have

$$\overline{(X_j\, x_{s+m})} = \overline{(X_j\, \phi_{s+m}(x_1, \ldots, x_s))}, \qquad \left(\begin{matrix} j = 1, \ldots, r \\ m = 1, \ldots, n-s \end{matrix}\right).$$

Conversely, if the system admits all the infinitesimal transformations, it will admit all the finite transformations of the group; for let $f(x_1, \ldots, x_n)$ be any function of the variables, then we have proved that $\overline{X_1}, \ldots, \overline{X_r}$ being the contracted operators of $X_1, \ldots, X_r$ with respect to the equations (1)

$$\overline{X_j f} = \overline{X_j} \cdot \bar{f},$$

and therefore

$$\overline{(e_1 X_1 + \ldots + e_r X_r) f} = (e_1 \overline{X_1} + \ldots + e_r \overline{X_r}) \overline{f},$$

and $\overline{(e_1 X_1 + \ldots + e_r X_r)^2 f} = (e_1 \overline{X_1} + \ldots + e_r \overline{X_r})^2 \overline{f}$, and so on; if then f is any function such that

$$\overline{X_1 f} = 0, \ldots, \overline{X_r f} = 0,$$

$$\overline{(e^{e_1 X_1 + \ldots + e_r X_r} f)} = \overline{f};$$

that is, an equation admitting the infinitesimal transformations will admit all the finite transformations of the group.

Suppose now that we are seeking an equation system admitting a given group, the points, whose coordinates satisfy these equations, must either be points of general position with regard to the group or points of special position. Suppose that they are points of order h, and that q is the number of unconnected operators in the group $X_1, \ldots, X_r$; if h is less than q the points are ones of special position; if h is equal to q they are points of general position, and h cannot be greater than q (§ 91). We say that the equation system is of order h.

We now take

$$(1) \quad x_{s+m} = \phi_{s+m}(x_1, \ldots, x_s), \qquad (m = 1, \ldots, n-s)$$

to be the *known* equations giving the loci of points of order h; and $\overline{X}_1, \ldots, \overline{X}_r$ to be the known contracted operators of the group with respect to these equations; and we take $\overline{X}_1, \ldots, \overline{X}_h$ to be the h unconnected operators of the contracted group.

Any equation system of order h must therefore by means of the equations (1) be reducible to an equation system in the variables $x_1, \ldots, x_s$; and in order to find such a system it is only necessary to find the equation systems admitting $\overline{X}_1, \ldots, \overline{X}_r$. This equation system being of order h cannot allow the points satisfying it to be special points with regard to the group $\overline{X}_1, \ldots, \overline{X}_r$; for were they so, they would be of order less than h, which is contrary to our supposition.

The problem is therefore reduced to this; we are given h unconnected operators $\overline{X}_1, \ldots, \overline{X}_h$ forming a complete system; and we have to find all the equation systems which admit these operators, and are yet such that the points satisfying these equations are not of special position with respect to $\overline{X}_1, \ldots, \overline{X}_h$.

§ 106. By a change of the variables we can take $\overline{X_1}, \ldots, \overline{X_h}$ to be respectively

$$\overline{X_1} = \xi_{11}\frac{\partial}{\partial x_1} + \ldots + \xi_{1h}\frac{\partial}{\partial x_h},$$
$$\vdots$$
$$\overline{X_h} = \xi_{h1}\frac{\partial}{\partial x_1} + \ldots + \xi_{hh}\frac{\partial}{\partial x_h},$$

where $\xi_{ji}, \ldots$ are functions of $x_1, \ldots, x_h$, and $(s-h)$ other variables which occur as parameters; and the equation system we are seeking must not make the determinant

$$\begin{vmatrix} \xi_{11}, & \cdot & \cdot & \xi_{1h} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \xi_{h1}, & \cdot & \cdot & \xi_{hh} \end{vmatrix}$$

zero.

Suppose that $f(x_1, \ldots, x_s) = 0$ is one equation of the system admitted, then

$$\xi_{11}\frac{\partial f}{\partial x_1} + \ldots + \xi_{1h}\frac{\partial f}{\partial x_h} = 0$$
$$\vdots$$
$$\xi_{h1}\frac{\partial f}{\partial x_1} + \ldots + \xi_{hh}\frac{\partial f}{\partial x_h} = 0\,;$$

and therefore, since the determinant is not zero, we must have

$$\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_h} = 0.$$

The required equation system can then be only a system of equations in the variables $x_{h+1}, \ldots, x_s$; that is, the system of equations can only connect the common integrals of

$$\overline{X_1}(f) = 0, \ldots, \overline{X_h}(f) = 0.$$

Example. Consider the group of the fourth order,

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},\quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

This group is transitive, and its matrix is

$$\begin{Vmatrix} 0, & -z, & y \\ z, & 0, & -x \\ -y, & x, & 0 \\ x, & y, & z \end{Vmatrix}.$$

The only values of x, y, z which cause the determinants of the second or lower orders to vanish are $x = y = z = 0$; and obviously there cannot be contracted operators to correspond to a discrete number of special points.

Forming the determinants of the third order, we see that the equation $x^2+y^2+z^2 = 0$ causes all of these determinants to vanish; this equation is therefore admitted by the group, and defines points of order two. The contracted operators with respect to this equation will therefore form a group in two variables, and will have two unconnected operators, and cannot therefore have any common invariants, so that the only equation admitted by the group is the equation

$$x^2+y^2+z^2 = 0.$$

Example. Consider the simply transitive group

$$(y^2+z^2-x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - 2xz\frac{\partial}{\partial z},$$

$$(x^2+z^2-y^2)\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial z},$$

$$(x^2+y^2-z^2)\frac{\partial}{\partial z} - 2xz\frac{\partial}{\partial x} - 2yz\frac{\partial}{\partial y}.$$

The matrix is seen to be $(x^2+y^2+z^2)^3$, and when we equate this to zero we see that all determinants of the second order vanish, so that the equation

$$z = i(x^2+y^2)^{\frac{1}{2}}$$

(where the symbol i denotes $\sqrt{-1}$) defines the locus of points of order one. This is the only invariant surface with respect to the group; to obtain the invariant curves with respect to the group we must find the integrals of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0,$$

since the contracted operator is

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The invariant curves are therefore

$$y = ax, \quad x^2+y^2+z^2 = 0,$$

where a is a variable parameter.

It must not be supposed that an invariant of the contracted operators is an invariant of the group itself; in transitive groups they never could be such: in this example $\frac{y}{x}$ is an invariant of the contracted operator, but for the given group it is only invariant on the surface $x^2+y^2+z^2=0$.

If we take the group of order ten which transforms minimum curves into minimum curves, we see that since it contains $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ one of the determinants of its matrix is unity, and therefore there are no *special* points with respect to this group; and because it is transitive, and without special points, it cannot have any invariant equation.

CHAPTER IX

PRIMITIVE AND STATIONARY GROUPS

§ 107. We have seen that for the group which transforms minimum curves into minimum curves there is no invariant surface, but, since it transforms the sphere

$$a(x^2+y^2+z^2)+2gx+2fy+2hz+d=0$$

into some other sphere, it has an invariant family of surfaces, viz. the spheres in three-dimensional space.

The theory explained in § 99 would show us that for any group whatever we could find invariant families of surfaces. One case of this general theory is of particular interest, viz. when the number of parameters in the surface is less than the number of variables. Following the usual phraseology, we shall call the parameters involved in the equation of any surface the coordinates of the surface.

When the number of the coordinates of a surface is less than the number of variables we may express its equations in the form

$$\phi_1(x_1, \ldots, x_n) = c_1, \ldots, \quad \phi_{n-q}(x_1, \ldots, x_n) = c_{n-q};$$

$c_1, \ldots, c_{n-q}$ will then be the coordinates of the surface; and, since a point on it has q degrees of freedom in its motion, we say that the surface is a q-way locus in n-dimensional space, or briefly a q-fold.

We suppose the forms of the functions $\phi_1, \ldots, \phi_{n-q}$ to be fixed; if for all values of the coordinates $c_1, \ldots, c_{n-q}$ of the q-fold, the q-fold admits the transformations of the group $X_1, \ldots, X_r$ the group must be intransitive. Since the q-folds can only each individually admit the group when $\phi_1, \ldots, \phi_{n-q}$ are invariants of the group, we see that the group cannot have more than q unconnected operators.

Suppose now that the group is intransitive, and that $x_{q+1}, \ldots, x_n$ are its invariants; we then have

$$X_k = \xi_{k1}(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \ldots + \xi_{kq}(x_1, \ldots, x_n)\frac{\partial}{\partial x_q}, \quad (k=1, \ldots, r).$$

The equations $x_{q+1}=a_{q+1}, \ldots, x_n = a_n$ are invariant for the group; suppose that $x_1, \ldots, x_q, a_{q+1}, \ldots, a_n$ is a point of general position, the contracted operators with respect to these equations are $\overline{X_1}, \ldots, \overline{X_r}$, where

$$\overline{X_k} = \xi_{k1}(x_1, \ldots, x_q, a_{q+1}, \ldots, a_n)\frac{\partial}{\partial x_1} + \ldots$$
$$+ \xi_{kq}(x_1, \ldots, x_q, a_{q+1}, \ldots, a_n)\frac{\partial}{\partial x_q}.$$

We know that these contracted operators will generate a group, and that q of its operators will be unconnected, so that this group, being in q variables, will be transitive.

If we say that the transformation

$$x_i' = e^{e_1\overline{X_1}+\ldots+e_r\overline{X_r}}x_i, \qquad (i = 1, \ldots, n)$$

in the group $\overline{X_1}, \ldots, \overline{X_r}$ corresponds to the transformation

$$x_i' = e^{e_1 X_1+\ldots+e_r X_r}x_i, \qquad (i = 1, \ldots, n)$$

in the group $X_1, \ldots, X_r$; then any point on the q-fold

$$x_{q+1} = a_{q+1}, \ldots, x_n = a_n$$

is transformed to the same point on that q-fold by either of these corresponding transformations.

Now the group $\overline{X_1}, \ldots, \overline{X_r}$ is transitive, and therefore any arbitrarily selected point on this q-fold can by the operations of this group be transformed to any other arbitrarily selected point on the q-fold: it follows that by the operations of the group $X_1, \ldots, X_r$ any point on this q-fold can be transformed to any other point on the same q-fold.

§ 108. Without, however, assuming that any one of the q-folds

$$\phi_1(x_1, \ldots, x_n) = c_1, \ldots, \phi_{n-q}(x_1, \ldots, x_n) = c_{n-q}$$

is transformed into itself by the operations of the group, we shall suppose that the totality of them is invariant; that is, the q-fold with the coordinates $c_1, \ldots, c_{n-q}$ is transformed to the q-fold with the coordinates $c_1', \ldots, c_{n-q}'$, the forms of the functions $\phi_1, \ldots, \phi_{n-q}$ which define the q-folds being of course fixed.

If $x_1, \ldots, x_n$ is a point on

$$\phi_1(x_1, \ldots, x_n) = c_1, \ldots, \phi_{n-q}(x, \ldots, x_n) = c_{n-q},$$

and if this point is transformed into $x_1', \ldots, x_n'$ then we must have

$$\phi_1(x_1', \ldots, x_n') = c_1', \ldots, \phi_{n-q}(x_1', \ldots, x_n') = c_{n-q}';$$

but unless the group is intransitive, and $\phi_1, \ldots, \phi_{n-q}$ are its invariants, we cannot have

$$\phi_1(x_1, \ldots, x_n) = \phi_1(x_1', \ldots, x_n'), \ldots,$$
$$\phi_{n-q}(x_1, \ldots, x_n) = \phi_{n-q}(x_1', \ldots, x_n').$$

If, however, the totality of q-folds is invariant we have, whether the group is intransitive or not, an invariant decomposition of space into ∞^{n-q} q-folds.

A group under which some decomposition of space is invariant is said to be *imprimitive*; a group under whose operations no such decomposition is possible is said to be *primitive*; thus intransitive groups are a particular class of imprimitive groups, and primitive groups are a particular class of transitive groups.

§ 109. Let

$$(1) \quad x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

be the equations of the given group, and let

$$\phi_1(x_1, \ldots, x_n) = c_1, \ldots, \phi_{n-q}(x_1, \ldots, x_n) = c_{n-q}$$

be an invariant decomposition of space; when we apply to this q-fold the transformation (1) we get

$$\phi_1(x_1', \ldots, x_n') = c_1', \ldots, \phi_{n-q}(x_1', \ldots, x_n') = c_{n-q}',$$

and we must therefore have an equation system of the form

$$c_i' = \psi_i(c_1, \ldots, c_{n-q}, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n-q).$$

It follows therefore from our first notions of a group that the functions $\psi_1, \ldots, \psi_{n-q}$ will define a group containing the identical transformation and r infinitesimal transformations, though these are not necessarily *independent*.

The variables in this group are the coordinates of the q-folds in space $x_1, \ldots, x_n$, and we may say that we have passed to a new space in $(n-q)$ dimensions; to any assigned point in this new space there will correspond a definite q-fold in the space $x_1, \ldots, x_n$; and to any transformation

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$$

in the original space there will correspond a transformation

$$c_i' = \psi_i(c_1, \ldots, c_{n-q}, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n-q)$$

in the new space.

By a change of the variables we may take

$$x_{q+1} = c_{q+1}, \ldots, x_n = c_n,$$

to be the equations of any q-fold, whose family is unaltered by the operations of the imprimitive group $X_1, \ldots, X_r$.

In this system of coordinates the finite equations of the imprimitive group must be of the form

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, q),$$
$$x_{q+j}' = f_{q+j}(x_{q+1}, \ldots, x_n, a_1, \ldots, a_r), \qquad (j = 1, \ldots, n-q);$$

for any q-fold of the system must by the operations of this group be transformed into some other.

The infinitesimal operators of the group are now

$$X_k = \xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, r),$$

where $\xi_{k,\,q+j}, \ldots$ do not involve $x_1, \ldots, x_q$.

It therefore follows from the identity

$$(X_i, X_k) = \sum^{h=r} c_{ikh} X_h$$

that the r operators $Z_1, \ldots, Z_r$, where

$$Z_k = \xi_{k,\,q+1}\frac{\partial}{\partial x_{q+1}} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, r),$$

form a group, such that

$$(Z_i, Z_k) = \sum^{h=r} c_{ikh} Z_h;$$

this group, however, is not necessarily of order r since the operators may not be *independent*.

§ 110. The complete system of equations

$$\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_q} = 0$$

is invariant under all the operations of the imprimitive group $X_1, \ldots, X_r$. This is at once seen to follow from the fact that $\xi_{k,q+j}, \ldots$ do not involve $x_1, \ldots, x_q$.

Conversely, if any complete system is invariant under the operations of a group, that group must be imprimitive. For by a change of coordinates we can take the complete system to be

$$\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_q} = 0,$$

and then, if

$$\xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}$$

is an operator of the group which the system admits, we see that $\xi_{k,\,q+j}, \ldots$ cannot involve $x_1, \ldots, x_q$; and therefore the equations

$$x_{q+1} = c_{q+1}, \ldots, x_n = c_n$$

can only be transformed to equations of the form

$$x_{q+1} = c'_{q+1}, \ldots, x_n = c'_n,$$

that is, the group is imprimitive.

§ 111. We have now seen that groups may be divided into transitive and intransitive classes of groups; and also into primitive and imprimitive classes; there is yet a third division into *stationary* and *non-stationary* groups. To explain this last division, let $X_1, \ldots, X_r$ be the r operators of the group where

$$X_k = \xi_{k1}(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}(x_1, \ldots, x_n)\frac{\partial}{\partial x_n}, \quad (k = 1, \ldots, r),$$

and suppose that exactly q of these operators are unconnected, say $X_1, \ldots, X_q$; and let

$$(1)\quad X_{q+j} = \sum^{k=q} \phi_{q+j,\,k}(x_1, \ldots, x_n)\, X_k, \qquad (j = 1, \ldots, r-q).$$

Let $x_1^0, \ldots, x_n^0$ be a point of general position, that is, a point such that not all q-rowed determinants in the matrix

$$\begin{vmatrix} \xi_{11}, & \cdot & \cdot & \cdot & \xi_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_{q1}, & \cdot & \cdot & \cdot & \xi_{qn} \end{vmatrix}$$

vanish, when the coordinates of this point are substituted in it. First we see that any infinitesimal transformation of the form

$$x'_i = x_i + t\,(e_1 X_1 + \ldots + e_q X_q)\, x_i, \qquad (i = 1, \ldots, n)$$

will transform the point $x_1^0, \ldots, x_n^0$ to some neighbouring point; for if the point remained fixed we should have

$$e_1 \xi_{1i}^0 + \ldots + e_q \xi_{qi}^0 = 0, \qquad (i = 1, \ldots, n),$$

and therefore all q-rowed determinants of the matrix would vanish.

The necessary and sufficient conditions that

$$e_1 X_1 + \ldots + e_r X_r$$

should not alter the point $x_1^0, \ldots, x_n^0$ are

$$e_1 \xi_{1i}^0 + \ldots + e_r \xi_{ri}^0 = 0, \qquad (i = 1, \ldots, n);$$

and these equations may by (1) be written in the form

$$\sum_{k=q} (e_k + \sum_{j=r-q} e_{q+j} \phi_{q+j,k}^0) \xi_{ki}^0 = 0, \qquad (i = 1, \ldots, n).$$

Since then the point $x_1^0, \ldots, x_n^0$ is one of general position, we must have

$$e_k + \sum^{j=r-q} e_{q+j} \phi_{q+j,k}^0 = 0, \qquad (k = 1, \ldots, q),$$

and the general form of an operator of the group which does not alter this point must be

$$\sum^{j=r-q} e_{q+j} (X_{q+j} - \sum^{k=q} \phi_{q+j,k} (x_1^0, \ldots, x_n^0) X_k).$$

It follows, since the transformations which leave a given point at rest must obviously have the group property, that the $(r-q)$ independent operators

$$X_{q+j} - \sum^{k=q} \phi_{q+j,k}^0 X_k, \qquad (j = 1, \ldots, r-q)$$

generate a sub-group.

We call this sub-group the group of the point $x_1^0, \ldots, x_n^0$. Unless all the operators of a group are unconnected, to each point of general position there will correspond one of these sub-groups.

§ 112. Let now $y_1^0, \ldots, y_n^0$ be any other point of general position, we now wish to see whether all those infinitesimal transformations of the group which leave $x_1^0, \ldots, x_n^0$ at rest have the property of also leaving $y_1^0, \ldots, y_n^0$ at rest; that is, whether the groups of the two points are the same.

If the groups of the two points are the same then for all values of the parameters $e_{q+1}, \ldots, e_r$

$$\sum^{j=r-q} e_{q+j} \left(X_{q+j} - \sum^{k=q} \phi_{q+j,k}(x_1^0, \ldots, x_n^0) X_k\right)$$
$$= \sum^{j=r-q} \epsilon_{q+j} \left(X_{q+j} - \sum^{k=q} \phi_{q+j,k}(y_1^0, \ldots, y_n^0) X_k\right),$$

where $\epsilon_{q+1}, \ldots, \epsilon_r$ is some other set of parameters not involving $x_1, \ldots, x_n$.

Since the operators $X_1, \ldots, X_r$ are independent, this can only be true if

$$e_{q+1} = \epsilon_{q+1}, \ldots, e_r = \epsilon_r,$$

and if further

$$\sum^{j=r-q} e_{q+j} \left(\phi_{q+j,k}(x_1^0, \ldots, x_n^0) - \phi_{q+j,k}(y_1^0, \ldots, y_n^0)\right) = 0.$$

Now $e_{q+1}, \ldots e_r$ are independent, so that we must have

$$\phi_{q+j,k}(x_1^0, \ldots, x_n^0) = \phi_{q+j,k}(y_1^0, \ldots, y_n^0), \qquad \begin{pmatrix} j = 1, \ldots, r-q \\ k = 1, \ldots, \quad q \end{pmatrix}$$

as the necessary and sufficient conditions that the groups of the points $x_1^0, \ldots, x_n^0$ and $y_1^0, \ldots, y_n^0$ may coincide.

§ 113. The sub-group which leaves $x_1^0, \ldots, x_n^0$ at rest will therefore leave at rest all points on the manifold

$$(1) \qquad \phi_{q+j,k}(x_1, \ldots, x_n) = \phi_{q+j,k}(x_1^0, \ldots x_n^0),$$
$$\begin{pmatrix} j = 1, \ldots, r-q \\ k = 1, \ldots, \quad q \end{pmatrix}.$$

Of the functions $\phi_{q+j,k}, \ldots$ not more than n can be unconnected; if n are unconnected only a discrete number of points will lie on this manifold; and we then say that the group $X_1, \ldots, X_r$ is *non-stationary*. If, however, fewer than n of the functions are unconnected, say s, then the equations (1) define an $(n-s)$-way locus; and the group of the point $x_1^0, \ldots, x_n^0$ leaves invariant the continuous $(n-s)$-way locus which passes through the point; in this case we say that the group $X_1, \ldots, X_r$ is *stationary*. The groups of all points on this locus are the same; we shall call this locus the *group locus* of any point on it.

If $\quad x_1' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n)$

is any transformation of the group $X_1, \ldots, X_r$, and $X_1', \ldots, X_r'$

are the operators obtained by replacing x_i by x'_i in $X_1, \ldots, X_r$, we know from the discussion in § 75 that $X'_1, \ldots, X'_r$ are an independent set of operators of the group $X_1, \ldots X_r$. Suppose that by this transformation the point $x_1^0, \ldots, x_n^0$ becomes the point $x_1^{\prime 0}, \ldots, x_n^{\prime 0}$; then,

$$e_1 X_1 + \ldots + e_r X_r$$

being an operator which leaves $x_1^0, \ldots, x_n^0$ at rest,

$$e_1 X'_1 + \ldots + e_r X'_r$$

will be an operator leaving $x_1^{\prime 0}, \ldots, x_n^{\prime 0}$ at rest; and the group of the point $x_1^0, \ldots, x_n^0$ is therefore transformed into the group of the point $x_1^{\prime 0}, \ldots, x_n^{\prime 0}$. If then the group is stationary, the $(n-s)$-way group locus through $x_1^0, \ldots, x_n^0$ is transformed to the $(n-s)$-way group locus through $x_1^{\prime 0}, \ldots, x_n^{\prime 0}$. It follows therefore that a stationary group is imprimitive, since the group loci are transformed *inter se*.

It should be noticed that not all imprimitive groups, nor even all intransitive groups, are stationary; primitive groups however, having no invariant decomposition of space, must be non-stationary.

§ 114. We shall now give an analytical proof of the theorem that the equations

$$(1) \quad \phi_{q+j,k}(x_1, \ldots, x_n) = c_{q+j,k}, \qquad \begin{pmatrix} j = 1, \ldots, r-q \\ k = 1, \ldots, \quad q \end{pmatrix}$$

define an invariant decomposition of space into ∞^s $(n-s)$-way loci, where s is the number of the functions $\phi_{q+j,k}, \ldots$ which are *unconnected*.

From the fundamental group property

$$(X_p, X_{q+j}) = \sum^{i=r} c_{p,q+j,i} X_i, \quad \begin{pmatrix} j = 1, \ldots, r-q; \\ k = 1, \ldots, \quad q; \end{pmatrix} p = 1, \ldots, r \Big),$$

and from the identity

$$(2) \quad X_{q+j} = \sum^{k=q} \phi_{q+j,k} X_k, \qquad (j = 1, \ldots, r-q),$$

we deduce that

$$\sum^{m=r,\, k=q} c_{pkm} \phi_{q+j,k} X_m + \sum^{k=q} (X_p \phi_{q+j,k}) X_k = \sum^{i=r} c_{p,q+j,i} X_i.$$

If we apply to this the identity (2) so as to eliminate the

operators $X_{q+1}, \dots, X_r$, we can equate the coefficients of $X_1, \dots, X_q$ on each side of this identity, for $X_1, \dots, X_q$ are by hypothesis unconnected; we thus obtain

$$X_p\,\phi_{q+j,m} = c_{p,q+j,m} + \sum^{i=r-q} c_{p,q+j,q+i}\,\phi_{q+i,m} - \sum^{k=q} c_{p,k,m}\,\phi_{q+j,k}$$
$$-\sum^{k=q,\,i=r-q} c_{p,k,q+i}\,\phi_{q+j,k}\,\phi_{q+i,m}.$$

It therefore follows that by the infinitesimal transformation

$$x_i' = x_i + tX_p\,x_i, \qquad (i = 1, \dots, n)$$

all the points which lie on any one of the $(n-s)$-way group loci (1) are so transformed as to be points lying on some one other of these loci.

We may perhaps see this more clearly if we throw (as we may by a change of coordinates) the equations

$$\phi_{q+j,k}(x_1, \dots, x_n) = c_{q+j,k}$$

into the forms

(3) $$x_1 = c_1, \dots, x_s = c_s.$$

What we have then proved is that by any infinitesimal operation of the group, and therefore by any finite operation of the group, the coordinates $x_1, \dots x_s$ are transformed into functions of $x_1, \dots, x_s$; and therefore the $(n-s)$-way locus (3) into the $(n-s)$-way locus

$$x_1 = \gamma_1, \dots, x_s = \gamma_s$$

where $\gamma_1, \dots, \gamma_s$ are functions of $c_1, \dots, c_s$ and the parameters of the group $X_1, \dots, X_r$.

§ 115. The functions $\phi_{j\mu}(x_1, \dots, x_n)$ have only been defined for the case $j > q$, $\mu \not> q$; it is convenient to complete the definition by saying that when these inequalities are not satisfied $\phi_{j\mu}(x_1, \dots, x_n)$ is to be taken as identically zero.

We now define a set of functions $\Pi_{ijk}, \dots$ as follows:

$$\Pi_{ijk} = c_{ijk} + \sum^{t=r-q} c_{i,j,q+t}\,\phi_{q+t,k} + \sum^{\mu=q} c_{\mu ik}\phi_{j\mu} + \sum^{\mu=q,\,t=r-q} c_{\mu,i,q+t}\,\phi_{j\mu}\,\phi_{q+t,k}.$$

If $j \not> q$,

$$\Pi_{ijk} = c_{ijk} + \sum^{t=r-q} c_{i,j,q+t}\,\phi_{q+t,k};$$

if $k > q$,

$$\Pi_{ijk} = c_{ijk} + \sum^{\mu=q} c_{\mu ik}\phi_{j\mu};$$

and if $j \ngtr q$ and $k > q$,

$$\Pi_{ijk} = c_{ijk}.$$

Since $c_{ijk} + c_{jik} = 0$ for all values of i, j, k we have

$$X_p \phi_{q+j,k} = \Pi_{p,q+j,k}.$$

Since $$X_{q+i} = \sum^{k=q} \phi_{q+i,k} X_k,$$

$$X_{q+i}\phi_{q+j,m} = \sum^{k=q} \phi_{q+i,k} X_k \phi_{q+j,m},$$

and therefore

$$\Pi_{q+i,q+j,m} = \sum^{k=q} \phi_{q+i,k} \Pi_{k,q+j,m}, \qquad \begin{pmatrix} i = 1, \ldots, r-q \\ j = 1, \ldots, r-q \\ m = 1, \ldots, \quad q \end{pmatrix};$$

these are identities, satisfied by the functions $\phi_{q+i,k}, \ldots$.

Again, since

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k = \sum^{k=q} (c_{ijk} + \sum^{t=r-q} c_{i,j,q+t}\phi_{q+t,k}) X_k,$$

we see that, $X_1, \ldots, X_q$ being the unconnected operators of the group,

$$(X_i, X_j) = \sum^{k=q} \Pi_{ijk} X_k, \qquad \binom{i = 1, \ldots, q}{j = 1, \ldots, q};$$

we therefore call the functions Π_{ijk}, when none of the integers i, j, k exceed q, the *structure functions* of the complete system $X_1, \ldots, X_q$.

The functions $\phi_{q+j,k}, \ldots$ we shall call the *stationary functions*, since they determine whether the group to which they refer is stationary or not.

116. Suppose that s of these stationary functions are unconnected; we can by a suitable choice of new variables bring them to such a form that they will be functions of the variables $x_1, \ldots, x_s$ only; and we can also express the variables $x_1, \ldots, x_s$ in terms of the stationary functions.

The equations

(1) $$x_1 = c_1, \ldots, x_s = c_s$$

now give a decomposition of space which is invariant under

the operations of the group $X_1, \ldots, X_r$; only if s is less than n can we say that the group is stationary; and only if s is less than n can we say that the equations give a decomposition of space at all.

The operators of the group are $X_1, \ldots, X_r$ where X_k is

$$\xi_{k1}\frac{\partial}{\partial x_1}+\ldots+\xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k=1, \ldots, r),$$

and $\xi_{k1}, \ldots, \xi_{ks}$ are functions of $x_1, \ldots, x_s$ only; for the $(n-s)$-way locus (1) must by any operation of the group be transformed to some other $(n-s)$-way locus of the same family. If therefore

$$Z_k=\xi_{k1}\frac{\partial}{\partial x_1}+\ldots+\xi_{ks}\frac{\partial}{\partial x_s}, \qquad (k=1, \ldots, r),$$

$Z_1, \ldots, Z_r$ will generate a group, such that

$$(Z_i, Z_j)=\sum^{k=r} c_{ijk} Z_k,$$

where the structure of the group $X_1, \ldots, X_r$ is given by

$$(X_i, X_j)=\sum^{k=r} c_{ijk} X_k.$$

The group $Z_1, \ldots, Z_r$ is not, however, necessarily of order r, for its operators may not be *independent*.

We can construct this group $Z_1, \ldots, Z_r$ *merely from a knowledge of the structure constants and the stationary functions* of the group $X_1, \ldots, X_r$.

For if the stationary functions are known it merely requires an algebraic process to bring them to such a form that they are functions of $x_1, \ldots, x_s$ only. We can then say that $x_1, \ldots, x_s$ are known functions of the stationary functions; and, since $X_i\phi_{q+j,k}=\Pi_{i,q+j,k}$, and $\Pi_{i,q+j,k}$ is known in terms of the stationary functions, we see that $X_i\phi_{q+j,k}$ is also known in terms of them. It follows that $X_i x_1, \ldots, X_i x_s$ are all known functions, that is, the coefficients of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}$ in $X_1, \ldots, X_r$ are all known; that is, the operators $Z_1, \ldots, Z_r$ are known when the structure constants and the stationary functions are known.

§ 117. We have seen that the operators of an intransitive group can be simplified when we know its invariants; what we are now about to show is how by a suitable choice of

new variables to simplify these operators, and at the same time to simplify the stationary functions $\phi_{q+j,k}(x_1, \ldots, x_n), \ldots$.

We so choose the variables that the stationary functions are functions of the variables $x_1, \ldots, x_s$ only.

Of the invariants of $X_1, \ldots, X_q$, the unconnected operators of the group, some may be functions of $x_1, \ldots, x_s$ only; if we suppose that there are m such invariants, we may so choose the variables that these are $x_1, \ldots, x_m$; and m is not greater than the lesser of the two integers $n-q$ and s.

Since the stationary functions are now functions of $x_1, \ldots, x_s$, and $x_1, \ldots, x_m$ are invariants of $X_1, \ldots, X_q$, we have

$$X_k = \xi_{k,m+1}\frac{\partial}{\partial x_{m+1}} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, q),$$

where $\xi_{k,m+1}, \ldots, \xi_{k,s}$ are functions of $x_1, \ldots, x_s$ only.

Any function of $x_1, \ldots, x_m$ is an invariant of $X_1, \ldots, X_q$, but there are $(n-q-m)$ other invariants, unconnected with these. Let $f(x_1, \ldots, x_n)$ be one of these other invariants; since by hypothesis $x_1, \ldots, x_m$ are the only unconnected invariants which are mere functions of $x_1, \ldots, x_s$, f cannot be connected with $x_1, \ldots, x_s$; we may therefore again so choose the variables that f will be x_n.

In this system of variables the stationary functions are still mere functions of $x_1, \ldots, x_s$, and $x_1, \ldots, x_m, x_n$ are invariants of the group.

There now remain $(n-q-m-1)$ invariants, unconnected with $x_1, \ldots, x_m$ and x_n; let $f(x_1, \ldots, x_n)$ be one of these, we next prove that it cannot be connected with $x_1, \ldots, x_s, x_n$.

Suppose, if possible, that it is a mere function of $x_1, \ldots, x_s, x_n$; then, since it is annihilated by $X_1, \ldots, X_q$, we must have

$$\xi_{k,m+1}\frac{\partial f}{\partial x_{m+1}} + \ldots + \xi_{ks}\frac{\partial f}{\partial x_s} = 0, \qquad (k = 1, \ldots, q),$$

for $\xi_{kn} = 0$, because x_n is by hypothesis an invariant.

Now $\xi_{k,m+1}, \ldots, \xi_{ks}$ do not contain x_n; and therefore, if a_n is any arbitrary parameter, $f(x_1, \ldots, x_s, a_n)$ will be annihilated by $X_1, \ldots, X_q$. As we have proved that no function of $x_1, \ldots, x_s$ can be so annihilated, unless it is a mere function of $x_1, \ldots, x_m$, we conclude that $f(x_1, \ldots, x_s, x_n)$ is a function of $x_1, \ldots, x_m$ and x_n only; that is, it is not one of the $(n-q-m-1)$ other invariants. We can therefore by a fresh choice of the variables take the function f to be x_{n-1}; and in these new variables the stationary functions will still be

mere functions of $x_1, \ldots, x_s$, and $x_1, \ldots, x_m$, x_n, x_{n-1} will be invariants.

Proceeding thus, we see that we may finally take the stationary functions to be functions of the variables $x_1, \ldots, x_s$ only, and may take the $(n-q)$ unconnected invariants of the group to be $x_1, \ldots, x_m, x_{q+m+1}, \ldots, x_n$.

In proving this we have implicitly proved the inequality $q+m \geqq s$.

When a group is brought to this form we say it is in *standard form*.

§ 118. The above is the general method of bringing a group into *standard form* when it is intransitive, stationary, and when some one at least of the invariants of the group is a function of the stationary functions; the modification when any one of these conditions is not satisfied is simple, and the labour of bringing the group to standard form is lessened.

Thus, if the group is transitive, $q = n$, and $m = 0$; to bring the group to standard form involves only the algebraic processes of selecting the stationary functions in terms of which the others can be expressed, and taking them as a new set of variables $x_1, \ldots, x_s$.

If $m = 0$ then $q \geqq s$, and the invariants may be taken to be $x_{q+1}, \ldots, x_n$, while the structure functions will involve $x_1, \ldots, x_s$ only.

If the group is non-stationary $s = n$ and $m = (n-q)$, and the invariants are $x_1, \ldots, x_{n-q}$, while the structure functions involve all the variables $x_1, \ldots, x_n$.

We saw in § 45 that in order to bring the equations of a group, given by its operators $X_1, \ldots, X_r$, to finite form it was necessary to find the invariants of

$$e_1 X_1 + \ldots + e_r X_r.$$

This problem is simplified for stationary groups; for, when we know the operators, we know the stationary functions, and can by algebraic processes bring the above operator to the form

$$\sum^{k=r,\, j=s} e_k\, \xi_{kj}(x_1, \ldots, x_s) \frac{\partial}{\partial x_j} + \sum^{k=r,\, t=n-s} e_k\, \xi_{k,\, s+t}(x_1, \ldots, x_n) \frac{\partial}{\partial x_{s+t}}.$$

There are $(s-1)$ unconnected invariants of this operator which are functions of $x_1, \ldots, x_s$; and these may be found by integration operations of order $(s-1)$: having found these, the remaining $(n-s)$ invariants may be found by integration operations of order $(n-s)$.

CHAPTER X

CONDITION THAT TWO GROUPS MAY BE SIMILAR. RECIPROCAL GROUPS

§ 119. The functions $\phi_{q+j,k}, \ldots$ which determine whether a given group is stationary or non-stationary are of much importance in other parts of group theory; we shall now consider their application to the problem of determining whether two assigned groups are or are not similar; that is, whether or not the one group can be transformed into the other, by a mere change of the variables.

Taking $X_1, \ldots, X_r$ to be the operators of a group of order r and $X_1, \ldots, X_q$ to be the unconnected operators of the group, we have

$$X_{q+j} = \sum^{k=q} \phi_{q+j,k}(x_1, \ldots, x_n) X_k, \qquad (j = 1, \ldots, r-q).$$

If we change to a new set of variables given by

$$y_i = f_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n),$$

the r operators $X_1, \ldots, X_r$ will be transformed into r independent operators $Y_1, \ldots, Y_r$, where

$$X_k = Y_k = \eta_{k1}\frac{\partial}{\partial y_1} + \ldots + \eta_{kn}\frac{\partial}{\partial y_n}, \qquad (k = 1, \ldots, r),$$

$\eta_{kj}, \ldots$ being functions of the variables $y_1, \ldots, y_n$.

At the same time the functions $\phi_{q+j,k}(x_1, \ldots, x_n), \ldots$ will be transformed into functions

$$\psi_{q+j,k}(y_1, \ldots, y_n), \ldots,$$

such that

$$\psi_{q+j,k}(y_1, \ldots, y_n) = \phi_{q+j,k}(x_1, \ldots, x_n), \qquad \binom{j = 1, \ldots, r-q}{k = 1, \ldots, q}.$$

We must have

$$(Y_i, Y_j) = \sum^{k=r} c_{ijk} Y_k,$$

since

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k, \text{ and } X_i = Y_i.$$

If then we have two groups, viz. $X_1, \ldots, X_r$ in the variables $x_1, \ldots, x_n$, and $Y_1, \ldots, Y_r$ in the variables $y_1, \ldots, y_n$, each group being of the r^{th} order, we see that these groups cannot be similar unless we can find a set of *independent* operators $Z_1, \ldots, Z_r$, *dependent* on the operators $Y_1, \ldots, Y_r$, and such that the structure constants of $Z_1, \ldots, Z_r$ are the same as those of the group $X_1, \ldots, X_r$; and also such that $Z_1, \ldots, Z_q$ are *unconnected*, and $Z_{q+1}, \ldots, Z_r$ *connected* with $Z_1, \ldots, Z_q$.

These conditions are necessary; suppose that they are fulfilled; we may then assume that the group $Y_1, \ldots, Y_r$ can be presented in such a form that the structure constants of $Y_1, \ldots, Y_r$ are the same as those of $X_1, \ldots, X_r$, that $Y_1, \ldots, Y_q$ are unconnected, and that $Y_{q+1}, \ldots, Y_r$ are given by

$$Y_{q+j} = \sum^{k=q} \psi_{q+j,k}(y_1, \ldots, y_n) Y_k, \qquad (j = 1, \ldots, r-q).$$

If the groups are to be similar we must further have

$$\phi_{q+j,k}(x_1, \ldots, x_n) = \psi_{q+j,k}(y_1, \ldots, y_n), \qquad \begin{pmatrix} j = 1, \ldots, r-q \\ k = 1, \ldots, \quad q \end{pmatrix}.$$

If from these equations we could deduce an equation between $x_1, \ldots, x_n$ alone or between $y_1, \ldots, y_n$ alone, it is clear that the groups could not be similar; it will now be proved that if no such relation can be deduced the groups are similar.

§ 120. Suppose that of these $q(r-q)$ functions

$$\phi_{q+j,k}(x_1, \ldots, x_n), \ldots$$

exactly s are unconnected, we know that $s \not> n$; between any $(s+1)$ of these functions there must be a functional equation; and therefore, since there is no equation connecting $y_1, \ldots, y_n$, there must be the same functional equation between the corresponding functions of $y_1, \ldots, y_n$.

It must be possible to find at least one transformation scheme

$$y_i' = f_i(y_1, \ldots, y_n), \qquad (i = 1, \ldots, n)$$

which will transform any s of the functions

$$\psi_{q+j,k}(y_1, \ldots, y_n), \ldots$$

into the respective forms

$$\phi_{q+j,k}(y_1', \ldots, y_n'), \ldots;$$

and therefore, since the same functional equation which connects any $(s+1)$ of the functions $\psi_{q+j,k}, \ldots$ will connect the

corresponding $(s+1)$ functions $\phi_{q+j,k}, \ldots$, we see that this transformation scheme will transform each of the functions $\psi_{q+j,k}(y_1, \ldots, y_n), \ldots$, into the corresponding function

$$\phi_{q+j,k}(y'_1, \ldots, y'_n), \ldots.$$

The theorem which is to be proved is therefore reduced to the following: $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ are two groups, each of order r, in the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively; the operators in the first group $X_1, \ldots, X_q$ are unconnected, and

$$X_{q+j} = \sum^{k=q} \phi_{q+j,k}(x_1, \ldots, x_n) X_k, \qquad (j = 1, \ldots, r-q);$$

in the second group $Y_1, \ldots, Y_q$ are unconnected, and

$$Y_{q+j} = \sum^{k=q} \phi_{q+j,k}(y_1, \ldots, y_n) Y_k, \qquad (j = 1, \ldots, r-q);$$

these groups will be similar if

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k,$$

and

$$(Y_i, Y_j) = \sum^{k=r} c_{ijk} X_k.$$

If by the transformation scheme

$$x'_i = f_i(x_1, \ldots, x_n), \qquad (i = 1, \ldots, n)$$

the stationary functions of $X_1, \ldots, X_r$ are brought to such a form that they are functions of $x_1, \ldots, x_s$ only, then the scheme

$$y'_i = f_i(y_1, \ldots, y_n), \qquad (i = 1, \ldots, n)$$

will make the stationary functions of $Y_1, \ldots, Y_r$ functions of $y_1, \ldots, y_s$ only.

From what we have proved in § 115 as to the form of the coefficients $\xi_{k1}, \ldots, \xi_{ks}$ in $X_1, \ldots, X_r$, we see that these coefficients will be the same functions of $x_1, \ldots, x_s$ that $\eta_{k1}, \ldots, \eta_{ks}$ are of $y_1, \ldots, y_s$; and therefore, if any function $f(x_1, \ldots, x_s)$ is an invariant of $X_1, \ldots, X_r$, $f(y_1, \ldots, y_s)$ will be an invariant of $Y_1, \ldots, Y_r$.

If therefore we reduce each group to its standard form we may take

$$x_1, \ldots, x_m, \; x_{q+m+1}, \ldots, x_n$$

to be the invariants of $X_1, \ldots, X_r$, and its stationary functions to be functions of $x_1, \ldots, x_s$ only; and we may take

$$y_1, \ldots, y_m, \; y_{q+m+1}, \ldots, y_n$$

to be the invariants of $Y_1, \ldots, Y_r$, and its stationary functions to be the same functions of $y_1, \ldots, y_s$, that the stationary functions of the first group are of $x_1, \ldots, x_s$.

§ 121. Let us now say that the q-fold in x space

$$(1) \quad x_1 = a_1, \ldots, x_m = a_m, \; x_{m+q+1} = a_{m+q+1}, \ldots, x_n = a_n$$

corresponds to the q-fold in y space

$$(2) \quad y_1 = a_1, \ldots, y_m = a_m, \; y_{m+q+1} = f_{m+q+1}, \ldots, y_n = f_n,$$

where $f_{m+q+1}, \ldots, f_n$ are any $(n-m-q)$ fixed functions of their arguments $a_1, \ldots, a_m, \; a_{m+q+1}, \ldots, a_n$, such that $a_{m+q+1}, \ldots, a_n$ can be expressed in terms of $a_1, \ldots, a_m$ and $y_{m+q+1}, \ldots, y_n$.

We have now established such a correspondence between the two q-way loci, that when one is known the other is known.

Under the operations of the group $X_1, \ldots, X_r$ all of these q-folds in x space are invariant; and if on one of these we select any point P by an operation of the group $X_1, \ldots, X_r$ P can be transformed to any other point on the same q-fold. Similarly the q-folds in y space are each separately invariant under the operations of the group $Y_1, \ldots, Y_r$; and by a suitable operation of this group any point on one of these q-folds can be transformed to any other point on the same q-fold.

We now wish to establish a correspondence between the points in two corresponding q-folds, one in the x space and one in the y space.

We take as the 'initial' point on (1) the point P whose coordinates $x_{m+1}, \ldots, x_{m+q}$ are all zero; and we take as the 'initial' point on (2), which is to correspond to P, the point Q whose coordinates are

$$y_{m+1} = 0, \ldots, y_s = 0, \;\; y_{s+1} = f_{s+1}, \ldots, y_{m+q} = f_{m+q}$$

(we proved in § 117 that $m+q \not< s$), where $f_{s+1}, \ldots, f_{m+q}$ are any fixed functions of their arguments,

$$a_1, \ldots, a_m, \;\; a_{m+q+1}, \ldots, a_n.$$

We have now established a correspondence between the 'initial' points on any two corresponding q-folds; we get the correspondence between the two spaces by the convention that the points obtained by operating on the coordinates of P

with $$e^{e_1 X_1 + \ldots + e_r X_r}$$

shall respectively correspond to the points obtained by operating on the coordinates of Q with

$$e^{e_1 Y_1 + \ldots + e_r Y_r}.$$

There are 'initial' points P lying on each of the q-folds in x space; to take P, a point on any one particular q-fold, would merely establish a correspondence between the points of that q-fold and the corresponding q-fold in y space; by taking initial points on each q-fold we have the complete correspondence between the two spaces.

It must now be proved that we have established a point-to-point correspondence between the two spaces; i.e. the doubt must be removed as to whether the operators

$$e^{e_1 X_1 + \ldots + e_r X_r} \quad \text{and} \quad e^{\epsilon_1 X_1 + \ldots + \epsilon_r X_r},$$

applied to the point P might give the same point in x space, whereas the operators

$$e^{e_1 Y_1 + \ldots + e_r Y_r} \quad \text{and} \quad e^{\epsilon_1 Y_1 + \ldots + \epsilon_r Y_r},$$

applied to the point Q might give two different points in y space.

If $$e^{e_1 X_1 + \ldots + e_r X_r} \quad \text{and} \quad e^{\epsilon_1 X_1 + \ldots + \epsilon_r X_r},$$

applied to P give the same point, then the operator

$$e^{-e_1 X_1 - \ldots - e_r X_r}\, e^{\epsilon_1 X_1 + \ldots + \epsilon_r X_r}$$

will not alter the coordinates of P at all; that is, this operator will belong to the group of P.

By the second fundamental theorem (§ 50)

$$e^{-e_1 X_1 - \ldots - e_r X_r}\, e^{\epsilon_1 X_1 + \ldots + \epsilon_r X_r} = e^{\lambda_1 X_1 + \ldots + \lambda_r X_r},$$

where $\lambda_1, \ldots, \lambda_r$ are constants, which are functions of

$$e_1, \ldots, e_r, \; \epsilon_1, \ldots, \epsilon_r,$$

and the structure constants of the group $X_1, \ldots, X_r$; and therefore, as these structure constants are the same for the group $Y_1, \ldots, Y_r$,

$$e^{-e_1 Y_1 - \ldots - e_r Y_r}\, e^{\epsilon_1 Y_1 + \ldots + \epsilon_r Y_r} = e^{\lambda_1 Y_1 + \ldots + \lambda_r Y_r}.$$

The doubt which we have suggested as to the unique correspondence will be removed when we prove that if

$$\lambda_1 X_1 + \ldots + \lambda_r X_r$$

is an operator of the group of the point P with respect to $X_1, \ldots, X_r$, then

$$\lambda_1 Y_1 + \ldots + \lambda_r Y_r$$

will be an operator of the group of the point Q with respect to $Y_1, \ldots, Y_r$.

Since $\lambda_1 X_1 + \ldots + \lambda_r X_r$ is an operator of the group of P, we have by § 111,

$$\lambda_k + \sum^{j=r-q} \lambda_{q+j}\, \phi_{q+j,k}(x_1^0, \ldots, x_n^0) = 0, \qquad (k = 1, \ldots, q),$$

where $x_1^0, \ldots, x_n^0$ are the coordinates of P.

Now by hypothesis the functions $\phi_{q+j,k}, \ldots$ only involve the coordinates $x_1, \ldots, x_s$; and if the coordinates of Q are $y_1^0, \ldots, y_n^0$, we have $y_1^0 = x_1^0, \ldots, y_s^0 = x_s^0$, so that

$$\lambda_k + \sum^{j=r-q} \lambda_{q+j}\, \phi_{q+j,k}(y_1^0, \ldots, y_n^0), \qquad (k = 1, \ldots, q);$$

and therefore $\lambda_1 Y_1 + \ldots + \lambda_r Y_r$ is an operator of the group of Q with respect to $Y_1, \ldots, Y_r$.

§ 122. We have therefore established a point-to-point correspondence between the two spaces; it may be noticed that, having proved that the coefficients of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}$ in $X_1, \ldots, X_r$ are the same functions of $x_1, \ldots, x_s$ that the corresponding coefficients of $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_s}$ in $Y_1, \ldots, Y_r$ are of $y_1, \ldots, y_s$, it will now follow that, if $y_1, \ldots, y_n$ is the point in y space which corresponds to $x_1, \ldots, x_n$ in x space, we must have

$$y_1 = x_1, \ldots, y_s = x_s.$$

Let S denote the transformation scheme which transforms any point $x_1, \ldots, x_n$ to the corresponding point $y_1, \ldots, y_n$ in the other space, then $Sf(x_1, \ldots, x_n)$ will be equal to $f(y_1, \ldots, y_n)$ where f is any function of its arguments.

We take P to be the 'initial' point on any q-fold in x space; by varying the coordinates of this q-fold, and the parameters $e_1, \ldots, e_r$ in the operator

$$e^{e_1 X_1 + \ldots + e_r X_r},$$

this operator applied to the coordinates of an initial point P will transform it to any point in space x.

We may say then that

$$e^{e_1 X_1 + \ldots + e_r X_r} P$$

will be a general expression for any point in the x space.

The point in the y space which corresponds to this will be

$$e^{e_1 Y_1 + \ldots + e_r Y_r} Q,$$

and therefore

$$S e^{e_1 X_1 + \ldots + e_r X_r} P = e^{e_1 Y_1 + \ldots + e_r Y_r} Q,$$

or, $$e^{-e_1 Y_1 - \ldots - e_r Y_r} S e^{e_1 X_1 + \ldots + e_r X_r} P = Q.$$

We now take another independent set of parameters $\epsilon_1, \ldots, \epsilon_r$, then

$$\begin{aligned} e^{\epsilon_1 Y_1 + \ldots + \epsilon_r Y_r} e^{-e_1 Y_1 - \ldots - e_r Y_r} S\, e^{e_1 X_1 + \ldots + e_r X_r} P & \\ &= e^{\epsilon_1 Y_1 + \ldots + \epsilon_r Y_r} Q \\ &= S e^{\epsilon_1 X_1 + \ldots + \epsilon_r X_r} P. \end{aligned}$$

Since $e^{e_1 X_1 + \ldots + e_r X_r} P$ is any point in the x space, we must then have the identity

$$\begin{aligned} e^{\epsilon_1 Y_1 + \ldots + \epsilon_r Y_r} e^{-e_1 Y_1 - \ldots - e_r Y_r} S & \\ &= S e^{\epsilon_1 X_1 + \ldots + \epsilon_r X_r} e^{-e_1 X_1 - \ldots - e_r X_r}; \end{aligned}$$

and by the second fundamental theorem we therefore have

$$e^{\lambda_1 Y_1 + \ldots + \lambda_r Y_r} S = S e^{\lambda_1 X_1 + \ldots + \lambda_r X_r},$$

where $\lambda_1, \ldots, \lambda_r$ are constants which are arbitrary, for they are functions of the structure constants, and the arbitrary constants $e_1, \ldots, e_r$ and $\epsilon_1, \ldots, \epsilon_r$.

Since we have now proved that

$$e^{\lambda_1 Y_1 + \ldots + \lambda_r Y_r} = S e^{\lambda_1 X_1 + \ldots + \lambda_r X_r} S^{-1},$$

we see that the groups are similar; and that they are transformed into one another by the transformation scheme S; and that the operators $X_1, \ldots, X_r$ are respectively transformed to $Y_1, \ldots, Y_r$.

§ 123. A very important theorem may almost immediately be deduced from the proof of the foregoing theorem on the similarity of groups; to obtain it, however, it is necessary to consider closely the form of the transformation scheme S, which has converted the points of the x space into the points of the y space.

This theorem is the answer to the question which now

arises, viz. what are the transformations which will transform each of the operators of a given group into itself?

We might put this question thus, what are the transformations which will transform

$$(1)\quad X_k = \xi_{k1}\frac{\partial}{\partial x_1} + \dots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \dots, r)$$

into

$$(2)\quad Y_k = \eta_{k1}\frac{\partial}{\partial y_1} + \dots + \eta_{kn}\frac{\partial}{\partial y_n}, \qquad (k = 1, \dots, r),$$

where $X_1, \dots, X_r$ are the operators of a group, and η_{ki} is the same function of $y_1, \dots, y_n$ that ξ_{ki} is of $x_1, \dots, x_n$?

Suppose that $X_1, \dots, X_s$ is in standard form; we take to correspond to the q-fold in x space given by

$$(3)\quad x_1 = a_1, \dots, x_m = a_m, \; x_{m+q+1} = a_{m+q+1}, \dots, x_n = a_n,$$

the q-fold in y space given by

$$(4)\quad y_1 = a_1, \dots, y_m = a_m, \; y_{m+q+1} = a_{m+q+1} + t_{m+q+1}, \dots,$$
$$y_n = a_n + t_n,$$

where $t_{s+1}, \dots, t_n$ are small constants which will not vary from q-fold to q-fold in space y.

To the 'initial' point P on (3) we take as correspondent on (4) a point Q, whose coordinates are

$$y_{m+1} = 0, \dots, y_s = 0, \; y_{s+1} = t_{s+1}, \dots, y_{m+q} = t_{m+q}.$$

If we now establish the correspondence between the two spaces we notice that the coordinates of Q differ infinitesimally from the coordinates of P. Therefore, since X_k is obtained by replacing the variables $y_1, \dots, y_n$ by $x_1, \dots, x_n$ respectively in Y_k, if P' is the point obtained by operating on P with any finite operator of the group $X_1, \dots, X_r$, and Q' the corresponding point obtained by operating on Q with the corresponding finite operator of the group $Y_1, \dots, Y_r$, the coordinates of P' will also differ infinitesimally from those of Q'.

We now have in this correspondence

$$y_1 = x_1, \dots, y_s = x_s,$$

and also, since $x_{m+q+1}, \dots, x_n$ are invariants,

$$y_{m+q+1} = x_{m+q+1} + t_{m+q+1}, \dots, y_n = x_n + t_n,$$

and finally

$$y_{s+j} = x_{s+j} + \sum^{i=m+q-s} t_{s+i}\,\zeta_{s+i,\,s+j}, \qquad (j = 1, \dots, m+q-s),$$

where $\zeta_{s+i,\,s+j}, \dots$ are some functions of the variables $x_1, \dots, x_n$.

These equations give $(n-s)$ infinitesimal transformations transforming (1) into (2); the corresponding linear operators are $Z_{s+1}, \ldots, Z_n$, where

$$Z_{m+q+i} = \frac{\partial}{\partial x_{m+q+i}} + \sum^{j=m+q-s} \zeta_{m+q+i,\, s+j} \frac{\partial}{\partial x_{s+j}}, \qquad (i = 1, \ldots, n-m-q),$$

$$Z_{s+i} = \sum^{j=m+q-s} \zeta_{s+i,\, s+j} \frac{\partial}{\partial x_{s+j}}, \qquad (i = 1, \ldots, m+q-s).$$

We shall now prove that the determinant

$$\begin{vmatrix} \zeta_{s+1,\, s+1}, & \cdot & \cdot & \zeta_{s+1,\, m+q} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \zeta_{m+q,\, s+1}, & \cdot & \cdot & \zeta_{m+q,\, m+q} \end{vmatrix}$$

does not vanish identically, and therefore conclude that these operators are unconnected.

When we take $x_{m+1} = 0, \ldots, x_{m+q} = 0$, that is, when we take $x_1, \ldots, x_n$ to be the point P, $y_1, \ldots, y_n$ will be the coordinates of the point Q, and therefore $y_{s+1} = t_{s+1}, \ldots, y_{m+q} = t_{m+q}$; it follows that $\zeta_{s+i,\, s+j}$ will then reduce to ϵ_{ij}, where, as usual, ϵ_{ij} is unity if $i = j$, and zero if $i \neq j$.

The determinant cannot then vanish identically, since it is equal to unity when we take $x_{m+1} = 0, \ldots, x_{m+q} = 0$.

Since any infinitesimal transformation which transforms (1) into (2) must transform y_1 into $x_1, \ldots, y_s$ into x_s, we see that there cannot be more than $(n-s)$ unconnected infinitesimal transformations which have the required property.

§ 124. We have now found $(n-s)$ unconnected operators $Z_{s+1}, \ldots, Z_n$ which have the property of leaving each of the operators $X_1, \ldots, X_r$ unaltered in form, and have proved that there is no operator unconnected with $Z_{s+1}, \ldots, Z_n$ which can have this property.

Applying the transformation

$$x_i' = x_i + tZ_k x_i, \qquad (i = 1, \ldots, n),$$

we see that

$$X_j' = X_j + t\,(Z_k, X_j), \qquad (j = 1, \ldots, r),$$

and therefore the alternant (Z_k, X_j) must vanish for $X_j' = X_j$.

The operators $Z_{s+1}, \ldots, Z_n$ form a complete system of which

the invariants are the stationary functions of $X_1, \ldots, X_r$; suppose now that

$$(Z_{s+i}, Z_{s+j}) = \sum^{k=n-s} \rho_{s+i, s+j, s+k} Z_{s+k},$$

where $\rho_{s+i, s+j, s+k}, \ldots$ are functions of $x_1, \ldots, x_n$.

Since X_m is permutable with Z_{s+i} and with Z_{s+j}, it follows from Jacobi's identity that it is permutable with the alternant (Z_{s+i}, Z_{s+j}); we therefore have

$$\sum^{k=n-s} (X_m \rho_{s+i, s+j, s+k}) Z_{s+k} = 0;$$

and therefore, since $Z_{s+1}, \ldots, Z_n$ are unconnected, each of the functions $\rho_{s+i, s+j, s+k}, \ldots$ is an invariant of the group

$$X_1, \ldots, X_r.$$

Suppose now that $X_1, \ldots, X_r$ is non-stationary; we see that there are no operators leaving the forms of the operators $X_1, \ldots, X_r$ unaltered; there are therefore no operators permutable with each of these operators.

If on the other hand $X_1, \ldots, X_r$ is stationary there are $(n-s)$ such operators, viz. $Z_{s+1}, \ldots, Z_n$; these will form a complete system

$$(Z_{s+i}, Z_{s+j}) = \sum^{k=n-s} \rho_{s+i, s+j, s+k} Z_{s+k},$$

of which the *structure functions* $\rho_{s+i, s+j, s+k}, \ldots$ are invariants of $X_1, \ldots, X_r$; if then $X_1, \ldots, X_r$ is a transitive group, these structure functions must be mere constants, and $Z_{s+1}, \ldots, Z_n$ will generate a group which will be finite and continuous, and have all of its operators unconnected.

§ 125. Suppose now that the group $X_1, \ldots, X_r$ is simply transitive; it is then stationary, for the stationary functions vanish identically; and in it $s = 0$ and $r = n$; it will now be proved that the simply transitive group $Z_1, \ldots, Z_n$ has the same structure as the group $X_1, \ldots, X_n$.

We may take as the n independent operators of $X_1, \ldots, X_n$

$$(1) \quad X_k = \frac{\partial}{\partial x_k} + \sum^{\mu=\nu=n} h_{k\mu\nu} x_\mu \frac{\partial}{\partial x_\nu} + \ldots, \qquad (k = 1, \ldots, n),$$

where the terms not written down are of the second or higher order in powers and products of $x_1, \ldots, x_n$.

We may similarly choose as the operators of $Z_1, \ldots, Z_n$

$$(2) \quad Z_k = -\frac{\partial}{\partial x_k} + \sum^{\mu=\nu=n} l_{k\mu\nu} x_\mu \frac{\partial}{\partial x_\nu} + \ldots, \qquad (k = 1, \ldots, n),$$

where $h_{k\mu\nu}, \ldots, l_{k\mu\nu}, \ldots$ are sets of constants.

Since $\quad (X_i, Z_k) = 0, \quad \binom{i = 1, \ldots, n}{k = 1, \ldots, n},$

we must have

$$\sum^{\nu=n} (l_{ki\nu} + h_{ik\nu}) \frac{\partial}{\partial x_\nu} + \ldots = 0,$$

where the terms omitted are of higher degree than those written down.

This identity gives

$$(3) \quad l_{ki\nu} + h_{ik\nu} = 0, \qquad \begin{pmatrix} i = 1, \ldots, n; & \nu = 1, \ldots, n \\ k = 1, \ldots, n; & \end{pmatrix}.$$

We also see that

$$(X_i, X_k) = \sum^{\nu=n} (h_{ki\nu} - h_{ik\nu}) \frac{\partial}{\partial x_\nu} + \ldots;$$

and therefore the structure constants of $X_1, \ldots, X_n$ are given by

$$c_{ik\nu} = h_{ki\nu} - h_{ik\nu}.$$

Similarly the structure constants of the group $Z_1, \ldots, Z_n$ are given by

$$c_{ikn} = l_{ik\nu} - l_{ki\nu};$$

and therefore by (3) we see that the two groups $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$ have the same structure constants when we take the independent operators in the respective forms (1) and (2).

The two groups $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_n$ are said to be *reciprocal* to one another.

CHAPTER XI

ISOMORPHISM

§ 126. We have proved in § 58 that the structure constants of a group are the same as those of its parameter group; we shall now give a second and more direct proof of this theorem.

If $$x_i' = e^{a_1 X_1 + \ldots + a_r X_r} x_i, \qquad (i = 1, \ldots, n)$$

are the canonical equations of a group, then we know that

$$(1) \quad e^{a_1 X_1 + \ldots + a_r X_r} e^{b_1 X_1 + \ldots + b_r X_r} = e^{c_1 X_1 + \ldots + c_r X_r},$$

where $c_1, \ldots, c_r$ are functions of $a_1, \ldots, a_r$, $b_1, \ldots, b_r$, and the structure functions of the group.

Let $$c_k = F_k(a_1, \ldots, a_r, \; b_1, \ldots, b_r), \qquad (k = 1, \ldots, r),$$

then $$y_k' = F_k(y_1, \ldots, y_r, \; a_1, \ldots, a_r), \qquad (k = 1, \ldots, r)$$

are the equations of the first parameter group in canonical form; and the equations of the second parameter group are

$$y_k' = F_k(a_1, \ldots, a_r, \; y_1, \ldots, y_r), \qquad (k = 1, \ldots, r).$$

The forms of the functions $F_1, \ldots, F_r$ are fixed by the identity (1), and can be determined in powers and products of $a_1, \ldots, a_r$, $b_1, \ldots, b_r$ when we merely know the structure constants of $X_1, \ldots, X_r$; the method of obtaining these functions is partly explained in Chapter IV, and more completely in a paper in the *Proceedings of the London Mathematical Society*, Vol. XXIX, 1897, pp. 14–32. As, however, we now only require the expansion up to and including powers of the second degree, we shall obtain this expansion from first principles.

Neglecting, then, all powers above the second, we have

$$e^{aX} e^{bY} = \left(1 + aX + \frac{a^2}{2} X^2\right)\left(1 + bY + \frac{b^2}{2} Y^2\right),$$

$$= 1 + aX + bY + \frac{a^2}{2} X^2 + abXY + \frac{b^2}{2} Y^2;$$

and therefore, since

$$(aX+bY)^2 = a^2X^2 + ab(XY+YX) + b^2Y^2,$$

$$e^{aX}e^{bY} = 1 + aX + bY + \tfrac{1}{2}(aX+bY)^2 + \tfrac{1}{2}ab(XY-YX).$$

This is true whatever the linear operators X and Y may be; and therefore the identity (1) gives

$$\begin{aligned}1 + c_1X_1 + \ldots + c_rX_r + \tfrac{1}{2}(c_1X_1 + \ldots + c_rX_r)^2 \\ = 1 + (a_1+b_1)X_1 + \ldots + (a_r+b_r)X_r \\ + \tfrac{1}{2}((a_1+b_1)X_1 + \ldots + (a_r+b_r)X_r)^2 \\ + \tfrac{1}{2}\sum^{j=i=r}(a_ib_j - a_jb_i)(X_i, X_j).\end{aligned}$$

To the first approximation we therefore have

$$c_k = a_k + b_k, \qquad (k = 1, \ldots, r).$$

In order to obtain the next approximation we substitute in the terms of the second degree $a_k + b_k$ for c_k, and, by aid of the identity

$$(X_i, X_j) = \sum^{k=r} c_{ijk}X_k,$$

we thus obtain

$$c_k = a_k + b_k + \tfrac{1}{2}\sum^{i=j=r}(a_ib_j - a_jb_i)c_{ijk} + \ldots.$$

From this we see that the first parameter group is

$$y'_k = y_k - a_k + \tfrac{1}{2}\sum^{i=j=r}(y_ia_j - y_ja_i)c_{ijk} + \ldots.$$

The identical transformation is obtained by taking

$$a_1 = 0, \ldots, a_n = 0;$$

and then

$$\frac{\partial y'_k}{\partial a_j} = \epsilon_{kj} + \tfrac{1}{2}\sum^{i=r} c_{ijk}y_i,$$

where ϵ_{kj} has its usual meaning.

§ 127. The infinitesimal operators of the first parameter group in canonical form are therefore

$$Y_j = \frac{\partial}{\partial y_j} + \tfrac{1}{2}\sum^{i=k=r} c_{ijk}y_i\frac{\partial}{\partial y_k} + \ldots, \qquad (j = 1, \ldots, r),$$

where the terms not written down are of higher degree in $y_1, \ldots, y_r$ than those written down.

Since $Y_1, \ldots, Y_r$ are the operators of a group we can, without any further calculation, find the structure constants of this group; for suppose that

$$(Y_i, Y_j) = \sum^{k=r} d_{ijk} Y_k,$$

we verify at once that $c_{ijk} = d_{ijk}$.

If we were to obtain the complete expansions for $Y_1, \ldots, Y_r$ we could verify the group property; and thus prove directly the third fundamental theorem, viz. that a simply transitive group can always be found to correspond to any assigned set of structure constants. All that we have attempted to prove, however, is that, $Y_1, \ldots, Y_r$ being known to generate a group, that group has the structure of the group $X_1, \ldots, X_r$.

Similarly we may see that the operators of the second parameter group in canonical form are

$$Z_j = \frac{\partial}{\partial y_j} - \tfrac{1}{2}\sum^{i=k=r} c_{ijk} y_i \frac{\partial}{\partial y_k} + \ldots, \qquad (j = 1, \ldots, r).$$

We know that these groups are simply transitive; and any operation of either is permutable with any operation of the other: they are therefore reciprocal groups, and we may easily verify that the structure constants of

$$Y_1, \ldots, Y_r \quad \text{and} \quad -Z_1, \ldots, -Z_r$$

are the same.

When we were given the finite equations of a group

$$x_i' = f_i(x_1, \ldots, x_n, \; a_1, \ldots, a_r), \qquad (i = 1, \ldots, n),$$

we found (§ 40) definite operators corresponding to the parameters $a_1, \ldots, a_r$, and we denoted these by

$$_aX_1, \ldots, {_aX_r}.$$

Any operator, however, dependent on these is equally an operator of the group; and when we are given any r independent operators $X_1, \ldots, X_r$ we can pass to another set $Y_1, \ldots, Y_r$, where

$$Y_k = h_{k1} X_1 + \ldots + h_{kr} X_r, \qquad (k = 1, \ldots, r),$$

and take these as the fundamental operators of the group, provided that the determinant

$$\begin{vmatrix} h_{11}, & \cdot & \cdot & \cdot & h_{1r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{r1}, & \cdot & \cdot & \cdot & h_{rr} \end{vmatrix}$$

does not vanish.

When therefore we speak of the canonical form of a group, we mean the canonical form corresponding to some one given set of operators $X_1, \ldots, X_r$. If we pass to a new set of operators we change the canonical form of the group; and therefore change the corresponding canonical forms of the parameter groups, by thus introducing a different set of structure constants.

§ 128. If we have two groups

$$(1)\ x'_i = e^{a_1 X_1 + \ldots + a_r X_r} x_i, \qquad (2)\ y'_i = e^{a_1 Y_1 + \ldots + a_r Y_r} y_i,$$

and if we denote by $S_{a_1, \ldots, a_r}$ that operation of the first which has the parameters $a_1, \ldots, a_r$, and by $T_{a_1, \ldots, a_r}$ the operation of the second with the same parameters, we say that $S_{a_1, \ldots, a_r}$ and $T_{a_1, \ldots, a_r}$ *correspond.*

It does not follow that, if $S_{a_1, \ldots, a_r}$ and $S_{b_1, \ldots, b_r}$ are two operations of the first group, and $T_{a_1, \ldots, a_r}$, $T_{b_1, \ldots, b_r}$ the corresponding operations of the second, the operation $S_{c_1, \ldots, c_r}$ will correspond to $T_{\gamma_1, \ldots, \gamma_r}$, where

$$S_{c_1, \ldots, c_r} = S_{a_1, \ldots, a_r} S_{b_1, \ldots, b_r}$$

and $$T_{\gamma_1, \ldots, \gamma_r} = T_{a_1, \ldots, a_r} T_{b_1, \ldots, b_r}.$$

This is only true if $\gamma_1 = c_1, \ldots, \gamma_r = c_r$; that is, if the two groups have the same parameter group.

Two groups are therefore then, and only then, simply isomorphic when they have the same parameter group.

Two groups, of which the fundamental set of operators of the first is $X_1, \ldots, X_r$, and of the second is $Y_1, \ldots, Y_r$ may not have, with respect to these operators, the same parameter group; and yet they may be thrown into such a form that they will have the same parameter group.

If we can find r *independent* operators, *dependent* on $Y_1, \ldots, Y_r$, and such that they have the same structure constants as $X_1, \ldots, X_r$, then, with respect to these new operators, the group $Y_1, \ldots, Y_r$ will have the same parameter group as $X_1, \ldots, X_r$.

Two groups of the same order

$$x'_i = e^{a_1 X_1 + \ldots + a_r X_r} x_i \quad \text{and} \quad y'_i = e^{a_1 Y_1 + \ldots + a_r Y_r} y_i,$$

are therefore then, and only then, simply isomorphic when the two sets of operators $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ have the same structure constants.

§ 129. Having explained what is meant when we say that two groups are simply isomorphic, we shall now consider the

analogous relation as to isomorphism of two groups whose orders are not the same.

Let (1) $$x'_i = e^{a_1 X_1 + \dots + a_r X_r} x_i$$

be a group of order r, and

(2) $$y'_i = e^{\alpha_1 Y_1 + \dots + \alpha_s Y_s} y_i$$

a group of order s, where $s < r$.

These groups may or may not be groups in the same number of variables; we establish a correspondence between the operations of the groups thus; we take

$$\alpha_k = h_{k1} a_1 + \dots + h_{kr} a_r, \qquad (k = 1, \dots, s),$$

where $h_{kj}, \dots$ are a set of constants such that not all s-rowed determinants vanish in the matrix

$$\left\| \begin{matrix} h_{11}, & . & . & . & h_{1r} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ h_{s1}, & . & . & . & h_{sr} \end{matrix} \right\|;$$

and we then say that the operation $T_{\alpha_1, \dots, \alpha_s}$ in the second corresponds to the operation $S_{a_1, \dots, a_r}$ in the first.

The first group is now said to be *multiply isomorphic* with the second, if the constants $h_{kj}, \dots$ can be so chosen that, whatever the values of the parameters $a_1, \dots, a_r, b_1, \dots, b_r$, the operation $T_{\alpha_1, \dots, \alpha_s} T_{\beta_1, \dots, \beta_s}$ corresponds to the operation $S_{a_1, \dots, a_r} S_{b_1, \dots, b_r}$, where β_k is the same function of $b_1, \dots, b_r$ that α_k is of $a_1, \dots, a_r$.

We know that $\alpha_1 = 0, \dots, \alpha_s = 0$ are the parameters of the identical transformation in (2); suppose that $a_1, \dots, a_r, b_1, \dots, b_r$ are two sets of values of parameters satisfying the equations

(3) $$0 = h_{k1} y_1 + \dots + h_{kr} y_r, \qquad (k = 1, \dots, s).$$

Since the identical transformation in (2) corresponds to $S_{a_1, \dots, a_r}$ and also to $S_{b_1 \dots, b_r}$, if the groups are isomorphic the identical transformation will also correspond to $S_{c_1, \dots, c_r}$, where $S_{c_1, \dots, c_r} = S_{a_1, \dots, a_r} S_{b_1, \dots, b_r}$, and therefore

$$0 = h_{k1} c_1 + \dots + h_{kr} c_r, \qquad (k = 1, \dots, s).$$

It follows that all the operations $S_{a_1, \dots, a_r}$ where $a_1, \dots, a_r$ are parameters satisfying the equation (3) form a sub-group of (1).

We shall next prove that this sub-group is self-conjugate.

Since (1) is in canonical form, the inverse operation to $S_{a_1, \ldots, a_r}$ is $S_{-a_1, \ldots, -a_r}$; that is,

$$S^{-1}{}_{a_1, \ldots, a_r} = S_{-a_1, \ldots, -a_r}.$$

Let $S_{b_1, \ldots, b_r}$ be any operation of (1), and $T_{\beta_1, \ldots, \beta_r}$ the corresponding operation of (2); then to $S^{-1}{}_{b_1, \ldots, b_r}$ there will correspond $T^{-1}{}_{\beta_1, \ldots, \beta_r}$ in (2). Therefore if $a_1, \ldots, a_r$ are the parameters of the sub-group the corresponding operation to $S_{b_1, \ldots, b_r}\ S_{a_1, \ldots, a_r}\ S^{-1}{}_{b_1, \ldots, b_r}$ must be the identical one; and therefore $S_{b_1, \ldots, b_r}\ S_{a_1, \ldots, a_r}\ S^{-1}{}_{b_1, \ldots, b_r}$ is itself an operation of this sub-group, and therefore the sub-group is a self-conjugate one.

§ 130. We may simplify the further discussion of the isomorphism of the two groups by taking $X_{s+1}, \ldots, X_r$ to be the operators of this self-conjugate sub-group. The equations (3) of § 129 must then be satisfied by $y_1 = 0, \ldots, y_s = 0$, and $y_{s+1}, \ldots, y_r$ may be taken arbitrarily: it follows that we must now have $h_{kj} = 0$ if $j > s$.

The equations which establish the correspondence between the operators of the two groups are now

$$\alpha_k = h_{k1}\,a_1 + \ldots + h_{ks}\,a_s, \qquad (k = 1, \ldots, s);$$

and it is easily seen that by taking a new set of operators, *dependent* on the first set $X_1, \ldots, X_s$, we may still further simplify these equations, and throw them into the form

$$\alpha_k = a_k, \qquad (k = 1, \ldots, s).$$

Since the first group is multiply isomorphic with the second,

$$e^{a_1 Y_1 + \ldots + a_s Y_s}\, e^{b_1 Y_1 + \ldots + b_s Y_s} \text{ and } e^{a_1 X_1 + \ldots + a_r X_r}\, e^{b_1 X_1 + \ldots + b_r X_r}$$

must correspond; and therefore, by considering the form of the functions $c_1, \ldots, c_r$ given in § 126, we can see that the structure constants of $Y_1, \ldots, Y_s$ are given by

$$(Y_i, Y_j) = \sum^{k=s} c_{ijk} Y_k, \qquad \binom{i = 1, \ldots, s}{j = 1, \ldots, s};$$

that is, the structure constants of $Y_1, \ldots, Y_s$ are the same as those of $X_1, \ldots, X_s$ if we only regard the coefficients of $X_1, \ldots, X_s$ and not those of $X_{s+1}, \ldots, X_r$ in the alternants

$$(X_i, X_j), \qquad \binom{i = 1, \ldots, s}{j = 1, \ldots, s}.$$

Unless, then, a group has a self-conjugate sub-group it cannot

be made multiply isomorphic with any group of lower order, except the group of zero order which consists merely of the identical transformation. A group which contains no self-conjugate group other than the group itself and the identical transformation is called a *simple* group, and therefore a simple group cannot be multiply isomorphic except with the identical transformation.

§ 131. *When we are given the structure constants of a group, we can find the structure constants of every group with which the first is multiply isomorphic.*

We shall see later on that, given the structure constants of a group, all the groups of such structure may be found; we now anticipate this result, and assume that, knowing the structure constants, we know the operators $X_1, \ldots, X_r$ of the group. There is no real need of the knowledge of these operators in the proof of the above theorem on isomorphism; it is, however, more simply expressed by aid of these operators.

Assuming, then, that we know the operators $X_1, \ldots, X_r$ we find a self-conjugate sub-group, and take its operators to be $X_{s+1}, \ldots, X_r$.

We now have

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k, \qquad \binom{i = 1, \ldots, s}{j = 1, \ldots, s},$$

and therefore

$$(X_m, (X_i, X_j)) = \sum^{k=s} c_{ijk} (X_m, X_k) + \sum^{t=r-s} c_{i,j,s+t} (X_m, X_{s+t}).$$

Since $X_{s+1}, \ldots, X_r$ is a self-conjugate sub-group, if we now apply Jacobi's identity to any three operators of the set $X_1, \ldots, X_s$ we can verify that

$$c_{ijk}, \ldots, \qquad \left(\begin{matrix} i = 1, \ldots, s; \\ j = 1, \ldots, s; \end{matrix} \quad k = 1, \ldots, s\right)$$

are a set of structure constants of order s.

If $Y_1, \ldots, Y_s$ is a group of order s with these structure constants, then $X_1, \ldots, X_r$ will be multiply isomorphic with $Y_1, \ldots, Y_s$; and in this way we obtain all groups with which $X_1, \ldots, X_r$ can be multiply isomorphic.

We may exhibit in a tabular form the relation of the two groups somewhat as in the Theory of Discontinuous Groups (Burnside, *Theory of Groups*, § 29).

If $e^{a_1 X_1 + \ldots + a_r X_r}$ is any finite operator of the group, of which $X_{s+1}, \ldots, X_r$ generate a self-conjugate sub-group, we

form a row containing this operator by allowing $a_1, \ldots, a_s$ to vary, and keeping $a_{s+1}, \ldots, a_r$ fixed; and we form the column containing this operator by allowing $a_{s+1}, \ldots, a_r$ to vary, and keeping $a_1, \ldots, a_s$ fixed.

If we take any row, and write in it $a_{s+1} = 0, \ldots, a_r = 0$, and replace X_1 by $Y_1, \ldots, X_s$ by Y_s, we have the finite operators of the second group; and to any two operators of the first group found in the same column only one operator in the second group will correspond.

§ 132. Suppose next that we are given a group $X_1, \ldots, X_r$ of order r such that

$$(X_i, X_j) = \sum^{k=r} c_{ijk} X_k,$$

and that we are also given r other operators $Y_1, \ldots, Y_r$ such that

$$(Y_i, Y_j) = \sum^{k=r} c_{ijk} Y_k;$$

and suppose further that only s of these operators are independent, viz. $Y_1, \ldots, Y_s$, and that

$$Y_{s+j} = h_{s+j,1} Y_1 + \ldots + h_{s+j,s} Y_s, \qquad (j = 1, \ldots, r-s).$$

If now instead of $X_1, \ldots, X_r$ we take any other set of independent operators $\overline{X_1}, \ldots, \overline{X_r}$, dependent on the first and such that
$$\overline{X_k} = l_{k1} X_1 + \ldots + l_{kr} X_r, \qquad (k = 1, \ldots, r);$$
and instead of $Y_1, \ldots, Y_r$ take $\overline{Y_1}, \ldots, \overline{Y_r}$ where

$$\overline{Y_k} = l_{k1} Y_1 + \ldots + l_{kr} Y_r,$$

then if

$$(1) \qquad (\overline{X_i}, \overline{X_j}) = \sum^{k=r} \bar{c}_{ijk} \overline{X_k},$$

we must also have

$$(2) \qquad (\overline{Y_i}, \overline{Y_j}) = \sum^{k=r} \bar{c}_{ijk} \overline{Y_k}.$$

It should be noticed that though from (1) we can infer (2), we could not infer (1) from (2).

We can now simplify the relation between the two sets of operators X and Y by taking as the independent operators of the group $\overline{X_1}, \ldots, \overline{X_r}$, where $\overline{X_1} = X_1, \ldots, \overline{X_s} = X_s$, and

$$\overline{X_{s+t}} = X_{s+t} - \sum^{k=s} h_{s+t,k} X_k, \qquad (t = 1, \ldots, r-s);$$

and we have

$$\overline{Y_1} = Y_1, \ldots, \overline{Y_s} = Y_s, \ \overline{Y_{s+t}} = 0, \qquad (t = 1, \ldots, r-s).$$

If $\bar{c}_{ijk}, \ldots$ are the structure constants with respect to $\overline{X_1}, \ldots, \overline{X_r}$ we now see (since $\overline{Y_{s+1}} \equiv 0$) that

$$\bar{c}_{s+i,j,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-s; \\ j = 1, \ldots, \quad r; \end{matrix} \ k = 1, \ldots, s\right),$$

and therefore $\overline{X_{s+1}}, \ldots, \overline{X_r}$ generate a self-conjugate group.

The operators $\overline{Y_1}, \ldots, \overline{Y_s}$ are now independent, and, since we have

$$(\overline{X_i}, \overline{X_j}) = \sum^{k=r} \bar{c}_{ijk} \overline{X_k}, \qquad \left(\begin{matrix} i = 1, \ldots, s \\ j = 1, \ldots, s \end{matrix}\right),$$

and

$$(\overline{Y_i}, \overline{Y_j}) = \sum^{k=s} \bar{c}_{ijk} \overline{Y_k}, \qquad \left(\begin{matrix} i = 1, \ldots, s \\ j = 1, \ldots, s \end{matrix}\right),$$

we see that $\overline{X_1}, \ldots, \overline{X_r}$ is multiply isomorphic with $\overline{Y_1}, \ldots, \overline{Y_s}$, the independent operators of the set $\overline{Y_1}, \ldots, \overline{Y_r}$; and that $\overline{X_{s+1}}, \ldots, \overline{X_r}$, the self-conjugate sub-group, corresponds to the identical transformation in the group of order s whose operators are $\overline{Y_1}, \ldots, \overline{Y_s}$.

§ 133. We had an example of isomorphic groups when we proved in § 104 that the contracted operators, with respect to any equation system which admitted the group $\overline{X_1}, \ldots, \overline{X_r}$, had the same structure constants as the operators $X_1, \ldots, X_r$. If the number of independent contracted operators is r, the isomorphism is simple; but if the number is less than r then $X_1, \ldots, X_r$ is multiply isomorphic with the group of its contracted operators.

Example. Prove that the group $X_1, \ldots, X_r$ is simply or multiply isomorphic with $E_1, \ldots, E_r$ where

$$E_k = \sum^{j=s=r} c_{jks} e_j \frac{\partial}{\partial e_s}, \qquad (k = 1, \ldots, r),$$

according as $X_1, \ldots, X_r$ does not, or does contain Abelian operators.

Example. Prove that if two transitive groups are simply isomorphic in such a way, that the sub-group of one, which leaves a point of general position at rest, corresponds to the sub-group in the other, which leaves the corresponding point

of general position at rest, then the two groups, if in the same number of variables, are similar.

The equations which define the groups of $x_1^0, \ldots, x_n^0$ and $y_1^0, \ldots, y_n^0$ are respectively (§ 111)

$$e_k + \sum^{j=r-n} e_{n+j}\, \phi_{n+j,k}(x_1^0, \ldots, x_n^0) = 0, \qquad (k = 1, \ldots, n),$$

and

$$\epsilon_k + \sum^{j=r-n} \epsilon_{n+j}\, \psi_{n+j,k}(y_1^0, \ldots, y_n^0) = 0, \qquad (k = 1, \ldots, n);$$

and therefore, since $e_i = \epsilon_i$, we must have

$$\phi_{n+j,k}(x_1^0, \ldots, x_n^0) = \psi_{n+j,k}(y_1^0, \ldots, y_n^0), \qquad \begin{pmatrix} j = 1, \ldots, r-n \\ k = 1, \ldots, \quad\; n \end{pmatrix}.$$

We have proved that

$$X_i\, \phi_{n+j,k} = \Pi_{i,n+j,k};$$

and therefore, if X_k^0 denotes the operator obtained from X_k by substituting for $x_1, \ldots, x_n$ the respective quantities $x_1^0, \ldots, x_n^0$, and $\phi_{n+j,k}^0$, $\Pi_{i,n+j,k}^0$ denote respectively the functions $\phi_{n+j,k}$, $\Pi_{i,n+j,k}$ with $x_1^0, \ldots, x_n^0$, substituted therein for $x_1, \ldots, x_n$, we have

$$X_i^0\, \phi_{n+j,k}^0 = \Pi_{i,n+j,k}^0.$$

Now since the two groups are simply isomorphic and $\phi_{n+j,k}^0 = \psi_{n+j,k}^0$, we must have

$$Y_i^0\, \psi_{n+j,k}^0 = X_i^0\, \phi_{n+j,k}^0, \qquad \begin{pmatrix} i = 1, \ldots, \quad\; r; \\ j = 1, \ldots, r-n; \end{pmatrix} k = 1, \ldots, n\Big);$$

and therefore, since

$$\phi_{n+j,k}(x_1, \ldots, x_n) = e^{e_1 X_1^0 + \ldots + e_r X_r^0}\, \phi_{n+j,k}(x_1^0, \ldots, x_n^0),$$

we must have

$$\phi_{n+j,k}(x_1, \ldots, x_n) = \psi_{n+j,k}(y_1, \ldots, y_n), \qquad \begin{pmatrix} j = 1, \ldots, r-n \\ k = 1, \ldots, \quad\; n \end{pmatrix}.$$

The groups therefore satisfy the sufficient and necessary conditions for similarity.

CHAPTER XII

ON THE CONSTRUCTION OF GROUPS WHOSE STRUCTURE CONSTANTS AND STATIONARY FUNCTIONS ARE KNOWN

§ 134. In Chapter X we proved that two groups are similar when they have the same structure constants and stationary functions. In this chapter we shall show how when these constants and functions are known the group may be constructed.

We take the case of transitive groups first; let $X_1, \ldots, X_n$ be unconnected and

$$(1)\quad X_{n+j} = \sum^{k=n} \phi_{n+j,k} X_k, \qquad (j = 1, \ldots, r-n);$$

suppose that s of the stationary functions are unconnected, and that these are functions of $x_1, \ldots, x_s$ only.

We saw (§ 115) that

$$(X_i, X_j) = \sum^{k=n} \Pi_{ijk} X_k, \qquad \binom{i = 1, \ldots, n}{j = 1, \ldots, n},$$

where $\Pi_{ijk}, \ldots$ are a known set of functions of $x_1, \ldots, x_s$ which we call the *structure functions* of the complete system $X_1, \ldots, X_n$; and if

$$X_k = \xi_{k1} \frac{\partial}{\partial x_1} + \ldots + \xi_{kn} \frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, n),$$

we proved that $\xi_{k1}, \ldots, \xi_{ks}$ are known functions of $x_1, \ldots, x_s$. It follows therefore that $X_m \Pi_{ijk}, \ldots$ are all known functions of $x_1, \ldots, x_s$.

The problem which lies before us is therefore to determine the forms of n unconnected operators in $x_1, \ldots, x_n$, such that

$$(X_i, X_j) = \sum^{k=n} \Pi_{ijk} X_k,$$

where the structure functions $\Pi_{ijk}, \ldots$ are known, and also

the functions obtained by operating on these functions with $X_1, \ldots, X_n$.

When we have found $X_1, \ldots, X_n$ then we shall also know $X_{n+1}, \ldots, X_r$ by (1).

If $s = n$, that is, if the group is non-stationary, since we know $\xi_{k1}, \ldots, \xi_{ks}$ we know $X_1, \ldots, X_n$ at once.

We now assume that $s < n$ so that the group is stationary.

§ 135. If we have any n unconnected operators we know (§ 68) that
$$(X_i, X_j) = \sum^{k=n} \rho_{ijk} X_k;$$
from the identities
$$(X_j, X_i) + (X_i, X_j) = 0,$$
$$\big(X_j, (X_i, X_k)\big) + \big(X_i, (X_k, X_j)\big) + \big(X_k, (X_j, X_i)\big) = 0,$$
we therefore deduce the following relations between the structure functions $\rho_{ijk}, \ldots$
$$(1) \qquad \rho_{ijk} + \rho_{jik} = 0,$$
$$X_j \rho_{ikm} + X_i \rho_{kjm} + X_k \rho_{jim} + \sum^{t=r} (\rho_{ikt}\rho_{jtm} + \rho_{kjt}\rho_{itm} + \rho_{jit}\rho_{ktm}) = 0,$$
where i, j, k, m may have any values from 1 to n.

If the structure functions $\rho_{ijk}, \ldots$ are mere constants $X_1, \ldots, X_n$ is a simply transitive group; and we have shown in Chapter V how from a knowledge of these constants the group itself may be constructed. In the case where $X_1, \ldots, X_n$ formed a group $X_m \rho_{jik}, \ldots$ were all zero; the problem before us now, when $\rho_{jik}, \ldots$ are known structure functions satisfying the conditions (1), and $X_m \rho_{jik}, \ldots$ are all known, but not necessarily zero, is to find the operators $X_1, \ldots, X_n$.

This problem is therefore a generalization of that considered in Chapter V, and we shall show how the results of Chapter V enable us to solve it.

Not more than n of the structure functions $\rho_{ijk}, \ldots$ can be unconnected; if n are unconnected we can express $x_1, \ldots, x_n$ in terms of these structure functions; and therefore, since we know $X_m \rho_{ijk}, \ldots$, we know $X_m(x_1), \ldots, X_m(x_n)$, and therefore know the operators $X_1, \ldots, X_n$.

We next suppose that only s are unconnected where $s < n$, and we may now assume that the variables have been so chosen that the structure functions only involve $x_1, \ldots, x_s$; if then
$$X_k = \xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n},$$

we see that $\xi_{k1}, \ldots, \xi_{ks}$ are all known functions of $x_1, \ldots, x_s$, and what we have to do is to determine $\xi_{k,s+1}, \ldots, \xi_{kn}$.

If we take

$$Y_k = \lambda_{k1} X_1 + \ldots + \lambda_{kn} X_n, \qquad (k = 1, \ldots, n),$$

where $\lambda_{kj}, \ldots$ are *known* functions of $x_1, \ldots, x_s$ whose determinant

$$\begin{vmatrix} \lambda_{11}, & . & . & . & \lambda_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \lambda_{n1}, & . & . & . & \lambda_{nn} \end{vmatrix}$$

does not vanish; then $Y_1, \ldots, Y_n$ will each be connected with $X_1, \ldots, X_n$ and they will form a complete system, so that

$$(Y_i, Y_j) = \sum^{k=n} \sigma_{ijk} Y_k.$$

The structure functions $\sigma_{ijk}, \ldots$ of this complete system must satisfy equations of condition like (1); they will be functions of $x_1, \ldots, x_s$ only, as will also be the functions $Y_m \sigma_{ijk}, \ldots$; and finally if we can construct the one set of operators we can construct the other set of operators.

We now make use of this principle to throw $X_1, \ldots, X_n$ into the forms

$$X_k \;= \frac{\partial}{\partial x_k} + \xi_{k,s+1} \frac{\partial}{\partial x_{s+1}} + \ldots + \xi_{kn} \frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, s),$$

$$X_{s+j} = \xi_{s+j,s+1} \frac{\partial}{\partial x_{s+1}} + \ldots + \xi_{s+j,n} \frac{\partial}{\partial x_n}, \qquad (j = 1, \ldots, n-s).$$

§ 136. In order to find the operators $X_1, \ldots, X_n$ which satisfy

$$(1) \qquad (X_i, X_j) = \sum^{k=n} \rho_{ijk} X_k, \qquad \binom{i = 1, \ldots, n}{j = 1, \ldots, n}$$

we have to find the set of functions $\xi_{ik}, \ldots$.

The only equations involving $\xi_{11}, \ldots, \xi_{1n}$, or such of them as are unknown, are those obtained by equating the coefficients of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ on each side of the identities

$$(X_j, X_k) = \rho_{jk1} X_1 + \ldots + \rho_{jkn} X_n, \qquad \binom{k = 1, \ldots, n}{j = 1, \ldots, n}.$$

We must therefore eliminate $\xi_{11}, \ldots, \xi_{1n}$ from

$$X_j \xi_{ki} - X_k \xi_{ji} = \sum^{m=n} \rho_{jkm} \xi_{mi}, \quad \binom{k = 1, \ldots, n; }{j = 1, \ldots, n;} \; i = 1, \ldots, n\Big),$$

and thus reduce the differential equations to be solved to a set not containing $\xi_{11}, \ldots, \xi_{1n}$.

In the form to which we have reduced $X_1, \ldots, X_n$ we see that $\rho_{ij1} = 0, \ldots, \rho_{ijs} = 0$; and thus we see that $\xi_{11}, \ldots, \xi_{1n}$ cannot appear in any of the identities, obtained by equating the coefficients of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ in (1), unless k or j is unity.

The only equations obtainable by differentiation and elimination from

$$(2) \quad X_k \xi_{1j} - X_1 \xi_{kj} = \sum^{m=n} \rho_{k1m} \xi_{mj}, \quad \binom{k = 1, \ldots, n}{j = 1, \ldots, n},$$

which will not involve derivatives of $\xi_{11}, \ldots, \xi_{1n}$ above the first, are

$$(3) \quad (X_i, X_k) \xi_{1j} - X_i X_1 \xi_{kj} + X_k X_1 \xi_{ij}$$
$$= X_i \sum^{m=n} \rho_{k1m} \xi_{mj} - X_k \sum^{m=n} \rho_{i1m} \xi_{mj}.$$

Now

$$X_i X_1 \xi_{kj} - X_k X_1 \xi_{ij} = X_1 (X_i \xi_{kj} - X_k \xi_{ij})$$
$$+ (X_i, X_1) \xi_{kj} - (X_k, X_1) \xi_{ij},$$

and

$$(4) \qquad X_i \xi_{kj} - X_k \xi_{ij} = \sum^{m=n} \rho_{ikm} \xi_{mj};$$

so that by aid of these equations and (1) we see that (3) takes the form

$$(5) \quad \begin{aligned} &\sum^{m=n} \rho_{ikm} (X_m \xi_{1j} - X_1 \xi_{mj}) - \sum^{m=n} \xi_{mj} (X_1 \rho_{ikm} + X_i \rho_{k1m} + X_k \rho_{1im}) \\ &+ \sum^{m=n} \rho_{1km} (X_i \xi_{mj} - X_m \xi_{ij}) + \sum^{m=n} \rho_{i1m} (X_k \xi_{mj} - X_m \xi_{kj}) = 0. \end{aligned}$$

We have, in passing to this form of (3), made use of the equations

$$\rho_{ijm} + \rho_{jim} = 0.$$

If we now replace

$$X_m \xi_{1j} - X_1 \xi_{mj} \text{ by } \sum^{p=n} \rho_{m1p} \xi_{pj},$$

and $$X_i \xi_{mj} - X_m \xi_{ij} \text{ by } \sum^{p=n} \rho_{imp} \xi_{pj},$$

the equation (5) is such that the coefficient of ξ_{pj} is seen to vanish identically by aid of the equations of condition (1) of § 135. We therefore conclude that the only equations of the first degree in the derivatives of $\xi_{11}, \ldots, \xi_{1n}$ are the equations (2) themselves. Any equation of the form (4) we shall denote symbolically by (i, k). What we have now proved is, that the only equations of the first degree in the derivatives of $\xi_{11}, \ldots, \xi_{1n}$ are the equations symbolized by

$$(1, 2), \ldots, (1, n).$$

§ 137. If then we have found any values of $\xi_{k1}, \ldots, \xi_{kn}$ (where k may have any value from 2 to n) to satisfy the equations

$$(1) \qquad (i, k), \qquad \left(\begin{matrix} i = 2, \ldots, n \\ k = 2, \ldots, n \end{matrix}\right),$$

the equations for $\xi_{11}, \ldots, \xi_{1n}$, viz. $(1, 2), \ldots, (1, n)$ will be consistent *.

By aid of these equations $(1, 2), \ldots, (1, n)$ we can express $X_2 \xi_{1j}, \ldots, X_n \xi_{1j}$ in terms of $\xi_{11}, \ldots, \xi_{1n}$ and known functions; for, assuming that we have solved the equations (1), $\xi_{k1}, \ldots, \xi_{kn}$ are known functions if $k > 1$.

Now $X_2, \ldots, X_n$ are $(n-1)$ unconnected operators, in which $\frac{\partial}{\partial x_1}$ does not occur; and, since $\xi_{k1}, \ldots, \xi_{kn}$, where $k > 1$, are known functions, these operators are known. We can therefore express $\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ in the forms

$$\frac{\partial}{\partial x_k} = \lambda_{k2} X_2 + \ldots + \lambda_{kn} X_n, \qquad (k = 2, \ldots, n),$$

where $\lambda_{kj}, \ldots$ are known functions of $x_1, \ldots, x_n$.

It follows therefore that, when we have solved the equations

* See a paper by the author on 'Simultaneous Equations' in the *Proceedings of the London Mathematical Society*, XXXI, p. 235.

(1), we can express the first derivatives of $\xi_{11}, \ldots, \xi_{1n}$ with respect to $x_2, \ldots, x_n$ in terms of $\xi_{11}, \ldots, \xi_{1n}$ and known functions; and in these expressions for the first derivatives $\xi_{11}, \ldots, \xi_{1n}$ will only occur linearly.

In these equations x_1 occurs merely as a parameter; we therefore look on x_1 as a constant, and say that we have obtained expressions for all the first derivatives of $\xi_{11}, \ldots, \xi_{1n}$ as linear functions of these unknowns, the coefficients being known functions of the variables; that is, the types of equations to be solved are

$$\frac{\partial u_j}{\partial x_k} = a_{jk1} u_1 + \ldots + a_{jkm} u_m + a_{jk, m+1}, \qquad \begin{pmatrix} j = 1, \ldots, m \\ k = 1, \ldots, n \end{pmatrix},$$

where a_{jki} are known functions of the variables; and of these equations integrals may be obtained in the form of power series.

The operators $X_{s+1}, \ldots, X_n$ form a complete system of order $(n-s)$, and the structure functions of this system only involve $x_1, \ldots, x_s$. Since these variables only enter the operators $X_{s+1}, \ldots, X_n$ as parameters we may look on the structure functions as mere constants; and we can therefore by the method of Chapter V find these operators $X_{s+1}, \ldots, X_n$.

$X_s, X_{s+1}, \ldots, X_n$ now form a complete system, and as we know $X_{s+1}, \ldots, X_n$ we may therefore by the method we have just described find the coefficients

$$\xi_{s, s+1}, \ldots, \xi_{sn},$$

and thus find the operator X_s.

Proceeding thus we may find all the operators $X_1, \ldots, X_n$, and have thus shown how a transitive group can be constructed when we know its structure constants and stationary functions.

§ 138. We can now construct the types of intransitive groups.

Let $X_1, \ldots, X_q$ be the unconnected operators of the group $X_1, \ldots, X_r$ which we suppose in standard form.

The stationary functions only involve $x_1, \ldots, x_s$, and, since $x_1, \ldots, x_m, x_{m+q+1}, \ldots, x_n$ are invariants,

$$X_k = \xi_{k, m+1} \frac{\partial}{\partial x_{m+1}} + \ldots + \xi_{k, m+q} \frac{\partial}{\partial x_{m+q}}, \qquad (k = 1, \ldots, r).$$

Since the invariants only enter $X_1, \ldots, X_q$ in the form of

parameters we may consider $X_1, \ldots, X_q$ to be the operators of a complete system in the q variables $x_{m+1}, \ldots, x_{m+q}$; and, as we have

$$(X_i, X_j) = \sum^{k=q} \Pi_{ijk} X_k, \qquad \begin{pmatrix} j = 1, \ldots, q \\ i = 1, \ldots, q \end{pmatrix},$$

where $\Pi_{ijk}, \ldots$ and $X_m \Pi_{ijk}, \ldots$ are known functions of the parameters $x_1, \ldots, x_m$ and the variables $x_{m+1}, \ldots, x_s$, we can construct the operators $X_1, \ldots, X_q$ as in the previous theory. When we have thus found $X_1, \ldots, X_q$ we can find the other operators by means of the identities

$$X_{q+j} = \sum^{k=q} \phi_{q+j,k} X_k, \qquad (j = 1, \ldots, r-q).$$

CHAPTER XIII

CONJUGATE SUB-GROUPS: THE CONSTRUCTION OF GROUPS FROM THEIR STRUCTURE CONSTANTS

§ 139. If $X_1, \ldots, X_r$ are the operators of a group with the structure constants $c_{ijk}, \ldots$ we have

$$c_{ijk} + c_{jik} = 0,$$

$$\sum_{}^{h=r} (c_{ikh}\, c_{hjm} + c_{kjh}\, c_{him} + c_{jih}\, c_{hkm}) = 0.$$

If $X_{q+1}, \ldots, X_r$ form a sub-group we also have

$$c_{q+i,\, q+k,\, h} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q\,; \\ k = 1, \ldots, r-q\,; \end{matrix}\; h = 1, \ldots, q\right);$$

and if this sub-group is self-conjugate we have the further conditions

$$c_{q+i,\, k,\, h} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q\,; \\ k = 1, \ldots, \quad r\,; \end{matrix}\; h = 1, \ldots, q\right).$$

Since our immediate object is to find the general form of a sub-group *conjugate* with a given sub-group, it will be convenient to take a set of operators $Y_1, \ldots, Y_r$ dependent on $X_1, \ldots, X_r$ and defined by

$$(1) \quad Y_k = X_k, \qquad (k = 1, \ldots, q),$$

$$(2) \quad Y_{q+t} = X_{q+t} - \sum^{\mu=q} h_{q+t,\mu}\, X_\mu, \qquad (t = 1, \ldots, r-q).$$

The identities (2) can be written

$$X_{q+t} = Y_{q+t} + \sum^{\mu=q} h_{q+t,\mu}\, Y_\mu\,;$$

and therefore, whatever values the constants $h_{q+t,\mu}, \ldots$ may have, $Y_1, \ldots, Y_r$ are independent operators.

If we suppose that $h_{i\mu} = 0$ when $i \not> q$, or when $\mu > q$, the formulae (1) and (2) may be replaced by

$$Y_i = X_i - \sum^{\mu=q} h_{i\mu} X_\mu, \qquad (i = 1, \ldots, r).$$

§ 140. We now introduce a set of functions of these constants $h_{q+t,\mu}, \ldots$ defined by

$$(1)\quad H_{ijk} = c_{ijk} + \sum^{t=r-q} c_{i,j,q+t}\, h_{q+t,k} + \sum^{\mu=q} c_{\mu ik}\, h_{j\mu} + \sum^{\mu=q} c_{j\mu k}\, h_{i\mu}$$

$$+ \sum^{\mu=\nu=q} c_{\mu\nu k}\, h_{i\mu}\, h_{j\nu} + \sum^{\mu=q,\, t=r-q} c_{\mu,i,q+t}\, h_{j\mu}\, h_{q+t,k} + \sum^{\mu=q,\, t=r-q} c_{j,\mu,q+t}\, h_{i\mu}\, h_{q+t,k}$$

$$+ \sum^{\mu=\nu=q,\, t=r-q} c_{\mu,\nu,q+t}\, h_{i\mu}\, h_{j\nu}\, h_{q+t,k}.$$

Since

$$(Y_i, Y_k) = (X_i, X_k) + \sum^{\mu=q} h_{i\mu}(X_k, X_\mu) + \sum^{\mu=q} h_{k\mu}(X_\mu, X_i)$$

$$+ \sum^{\mu=\nu=q} h_{i\mu}\, h_{k\nu}(X_\mu, X_\nu),$$

and $$(X_\mu, X_\nu) = \sum^{\lambda=r} c_{\mu\nu\lambda} X_\lambda = \sum^{\lambda=r} c_{\mu\nu\lambda}\Big(Y_\lambda + \sum^{j=q} h_{\lambda j} Y_j\Big),$$

we see that the structure constants of $Y_1, \ldots, Y_r$ are the set $H_{ijk}, \ldots$.

It therefore follows that

$$(2)\quad \begin{aligned} &H_{ijk} + H_{jik} = 0;\\ &\sum^{\mu=r}(H_{ik\mu}\, H_{\mu js} + H_{kj\mu}\, H_{\mu is} + H_{jim}\, H_{\mu ks}) = 0. \end{aligned}$$

Since $X_1, \ldots, X_r$ are derived from $Y_1, \ldots, Y_r$ by the law

$$X_i = Y_i + \sum^{\mu=q} h_{i\mu} Y_\mu, \qquad (i = 1, \ldots, r),$$

and $H_{ijk}, \ldots$ are the structure constants of $Y_1, \ldots, Y_r$, we must have

$$(3)\quad c_{ijk} = H_{ijk} - \sum^{t=r-q} H_{i,j,q+t}\, h_{q+t,k} - \sum^{\mu=q} H_{\mu ik}\, h_{j\mu} - \sum^{\mu=q} H_{j\mu k}\, h_{i\mu}$$

$$+ \sum^{t=r-q,\mu=q} H_{\mu,i,q+t}\, h_{j\mu}\, h_{q+t,k} + \sum^{t=r-q,\mu=q} H_{j,\mu,q+t}\, h_{i\mu}\, h_{q+t,k}$$

$$+ \sum^{\mu=\nu=q} H_{\mu\nu k}\, h_{i\mu}\, h_{j\nu} - \sum^{\mu=\nu=q,\, t=r-q} H_{\mu,\nu,q+t}\, h_{i\mu}\, h_{j\nu}\, h_{q+t,k}.$$

Let

$$(4)\quad \Pi_{ijk} = c_{ijk} + \sum^{t=r-q} c_{i,j,q+t}\, h_{q+t,k} + \sum^{\mu=q} c_{\mu ik}\, h_{j\mu} + \sum^{\mu=q,\, t=r-q} c_{\mu,i,q+t}\, h_{j\mu}\, h_{q+t,k},$$

then we see that

$$(5)\qquad H_{ijk} = \Pi_{ijk} - \sum^{\mu=q} h_{i\mu}\, \Pi_{\mu jk},$$

and therefore, since $h_{i\mu} = 0$ if $i \not> q$, $H_{ijk} = \Pi_{ijk}$ if $i \not> q$, and (5) can be replaced by

$$(6)\qquad \Pi_{ijk} = H_{ijk} + \sum^{\mu=q} h_{i\mu}\, H_{\mu jk}.$$

It will be noticed that though $H_{ijk} + H_{jik} = 0$, $\Pi_{ijk} + \Pi_{jik}$ is not zero if either i or j exceeds q.

If $k > q$, H_{ijk} takes the simpler form

$$(7)$$
$$H_{i,j,q+t} = c_{i,j,q+t} + \sum^{\mu=q} c_{\mu,i,q+t} h_{j\mu} + \sum^{\mu=q} c_{j,\mu,q+t} h_{i\mu} + \sum^{\mu=\nu=q} c_{\mu,\nu,q+t} h_{i\mu} h_{j\nu}.$$

§ 141. It is now necessary to prove the formula

$$(1)\quad \sum^{t=r} (\Pi_{\mu,q+j,t}\, \Pi_{\nu tk} - \Pi_{\nu,q+j,t}\, \Pi_{\mu tk}) = \sum^{t=r} c_{\nu\mu t}\, \Pi_{t,q+j,k}.$$

From (2) of the last article we see that

$$\sum^{t=r} (H_{\mu,q+j,t}\, H_{\nu tk} - H_{\nu,q+j,t}\, H_{\mu tk}) = \sum^{t=r} H_{\nu\mu t}\, H_{t,q+j,k}.$$

If we apply the formula (6) of § 140, we see that

$$\sum^{t=r} (\Pi_{\mu,q+j,t}\, \Pi_{\nu tk} - \Pi_{\nu,q+j,t} + \Pi_{\mu tk})$$
$$= \sum^{t=r} (H_{\mu,q+j,t} + \sum^{p=q} h_{\mu p}\, H_{p,q+j,t})\, (H_{\nu tk} + \sum^{p=q} h_{\nu p}\, H_{ptk})$$
$$- \sum^{t=r} (H_{\nu,q+j,t} + \sum^{p=q} h_{\nu p}\, H_{p,q+j,t})\, (H_{\mu tk} + \sum^{p=q} h_{\mu p}\, H_{ptk}).$$

Multiplying this out and applying (2) of § 140, we see that it is equal to

$$\sum^{t=r} H_{\nu\mu t}\, H_{t,q+j,k} + \sum^{t=r,\, p=q} h_{\mu p}\, H_{\nu pt}\, H_{t,q+j,k} + \sum^{t=r,\, p=q} h_{\nu p}\, H_{p\mu t}\, H_{t,q+j,k}$$
$$+ \sum^{t=r,\ p=p'=q} h_{\mu p}\, h_{\nu p'}\, H_{p'pt}\, H_{t,q+j,k}.$$

We now replace $H_{t,q+j,k}$ in this expression by

$$\Pi_{t,q+j,k}-\sum^{i=q} h_{ti}\,\Pi_{i,q+j,k},$$

and we see that, if $i \not> q$, the coefficient of $\Pi_{i,q+j,k}$ is the expression for $c_{\nu\mu i}$ in terms of $h_{q+t,k}, \ldots$ and the functions $H_{ijk}, \ldots$ given in (3) of § 140.

If $i > q$ this coefficient is

$$H_{\nu\mu i}+\sum^{p=q} h_{\mu p}\,H_{\nu pi}+\sum^{p=q} h_{\nu p}\,H_{p\mu i}+\sum^{p'=p=q} h_{\mu p}\,h_{\nu p'}\,H_{p'pi};$$

and if we notice that h_{ji} is zero when $i > q$, we shall see that this is also equal to $c_{\nu\mu i}$. We have thus verified the formula (1).

§ 142. We now look on $h_{q+t,k}, \ldots$ as a set of variable parameters; since every term which occurs in Π_{ijk} either begins with j or ends with k, we see that, if $j > q$ and $k \not> q$,

$$\frac{\partial \Pi_{ijk}}{\partial h_{j\mu}} = -\Pi_{i\mu k} \quad\text{and}\quad \frac{\partial \Pi_{ijk}}{\partial h_{q+t,k}} = \Pi_{i,j,q+t}.$$

We now introduce a set of r linear operators $\Pi_1, \ldots, \Pi_r$ defined by

$$\Pi_\mu = \sum^{k=q,\,j=r-q} \Pi_{\mu,q+j,k}\,\frac{\partial}{\partial h_{q+j,k}};$$

when we have

$$\Pi_\nu\,\Pi_{\mu,q+j,k} = -\sum^{t=q} \Pi_{\nu,q+j,t}\,\Pi_{\mu tk}+\sum^{t=r-q} \Pi_{\nu,q+t,k}\,\Pi_{\mu,q+j,q+t},$$

$$\Pi_\mu\,\Pi_{\nu,q+j,k} = -\sum^{t=q} \Pi_{\mu,q+j,t}\,\Pi_{\nu tk}+\sum^{t=r-q} \Pi_{\mu,q+t,k}\,\Pi_{\nu,q+j,q+t};$$

and therefore

$$\Pi_\nu\,\Pi_{\mu,q+j,k}-\Pi_\mu\,\Pi_{\nu,q+j,k} = -\sum^{t=r} \Pi_{\nu,q+j,t}\,\Pi_{\mu tk}+\sum^{t=r} \Pi_{\mu,q+j,t}\,\Pi_{\nu tk}$$

$$=\sum^{t=r} c_{\nu\mu t}\,\Pi_{t,q+j,k},$$

by the identity (1) of § 141.

It therefore follows that

$$(\Pi_i, \Pi_j) = \sum^{k=r} c_{ijk}\,\Pi_k, \qquad \begin{pmatrix} i = 1, \ldots, r \\ j = 1, \ldots, r \end{pmatrix},$$

so that $\Pi_1, \ldots, \Pi_r$ generate a group isomorphic with $X_1, \ldots, X_r$. If the operators $\Pi_1, \ldots, \Pi_r$ are independent the groups are simply isomorphic, but if they are not all independent $X_1, \ldots, X_r$ is multiply isomorphic with $\Pi_1, \ldots, \Pi_r$.

§ 143. Still looking on $h_{q+t,k}, \ldots$ as variables, we shall now prove that the equation system

$$(1)\quad H_{q+i,q+j,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q\,; \\ j = 1, \ldots, r-q\,; \end{matrix}\; k = 1, \ldots, q\right)$$

admits these operators.

If we notice that in $H_{q+i,q+j,k}$ every term either ends in k or begins with $q+i$ or $q+j$, we shall see that if $\mu \ngtr q$

$$\begin{aligned} \Pi_\mu H_{q+i,q+j,k} &= \sum^{p=q} H_{\mu,q+i,p} H_{q+j,p,k} + \sum^{p=q} H_{\mu,q+j,p} H_{p,q+i,k} \\ &\qquad + \sum^{t=r-q} H_{\mu,q+t,k} H_{q+i,q+j,q+t} \\ &= \sum^{p=r} (H_{q+i,\mu,p} H_{p,q+j,k} + H_{\mu,q+j,p} H_{p,q+i,k} \\ &\qquad + H_{q+j,q+i,p} H_{p\mu k}) \\ &\quad + \sum^{t=r-q} H_{\mu,q+t,k} H_{q+i,q+j,q+t} - \sum_{p=q+1}^{p=r} H_{q+i,\mu,p} H_{p,q+j,k} \\ &\quad - \sum_{p=q+1}^{p=r} H_{\mu,q+j,p} H_{p,q+i,k} - \sum^{p=r} H_{q+i,q+j,p} H_{\mu,p,k}. \end{aligned}$$

Since the expression in the bracket vanishes identically we see that $\Pi_\mu H_{q+i,q+j,k} = 0$ is an equation *connected* with the equation system (1); that is, it is satisfied for all values of the variables which satisfy (1).

Also since

$$H_{q+i,q+j,k} = \Pi_{q+i,q+j,k} - \sum^{\mu=q} h_{q+i,\mu} \Pi_{\mu,q+j,k},$$

we conclude that, even when $\mu > q$, the equation

$$\Pi \;\; H_{q+i,q+j,k} = 0$$

is connected with the equation system (1); so that we have proved that the system admits the operators $\Pi_1, \ldots, \Pi_r$.

It will be noticed that the operators $\Pi_1, \ldots, \Pi_r$ are defined simply from the structure constants $c_{ijk}, \ldots$ of the group, as are also the equations of the system (1) which admit these

operators. The group property of the operators $\Pi_1, \ldots, \Pi_r$ might have been proved without any reference to the group $X_1, \ldots, X_r$, though the labour of the proof was much lightened by that reference.

§ 144. Suppose now that we have any sub-group of $X_1, \ldots, X_r$ whose order is $(r-q)$, and suppose that all its operators are independent of $X_1, \ldots, X_q$; we may throw the operators of this sub-group into the form $Y_{q+1}, \ldots, Y_r$, where

$$Y_{q+t} = X_{q+t} - \sum^{\mu=q} h_{q+t,\mu} X_\mu, \qquad (t = 1, \ldots, r-q),$$

and we may then take $Y_1, \ldots, Y_r$ to be a set of r independent operators of the given group where $Y_k = X_k$ if $k \not> q$.

Since $H_{ijk}, \ldots$ are the structure constants of $Y_1, \ldots, Y_r$, and $Y_{q+1}, \ldots, Y_r$ is a sub-group,

$$H_{q+i,q+j,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, r-q; \end{matrix} \; k = 1, \ldots, q\right).$$

These are therefore the equations in the variable parameters $h_{q+t,k}, \ldots$ which define sub-groups of order $(r-q)$.

$Y_{q+1}, \ldots, Y_r$ will be a self-conjugate sub-group if

$$H_{j,q+i,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, \quad r; \end{matrix} \; k = 1, \ldots, q\right);$$

that is, the sub-group will then be invariant under any operation of the group $Y_1, \ldots, Y_r$.

Even when not invariant under *all* the operations of $Y_1, \ldots, Y_r$, that is, when not self-conjugate, it may be invariant under some of the operators.

It will be invariant under the operations of the sub-group $Y_{q+1}, \ldots, Y_r$ in every case; it will be invariant under the operations

$$x_i' = e^{a_{q-h} Y_{q-h} + \ldots + a_r Y_r} x_i, \qquad (i = 1, \ldots, n)$$

if, and only if,

$$H_{j,q+i,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q; \\ j = q-h, \ldots, r; \end{matrix} \; k = 1, \ldots, q\right).$$

The operations which transform a sub-group into itself must from first principles generate a group, which will contain the given sub-group as a sub-group, and therefore the operators $Y_{q-h}, \ldots, Y_r$ must themselves be a sub-group of $Y_1, \ldots, Y_r$.

§ 145. Suppose now that we are given the structure constants $c_{ijk}, \ldots$ of a group $X_1, \ldots, X_r$, and we want to find the structure constants of all possible sub-groups of order $(r-q)$; we equate to zero the functions $H_{q+i,q+j,k}, \ldots$ of the variables $h_{q+t,\mu}, \ldots$.

If no values of $h_{q+t,\mu}, \ldots$ can be found to satisfy the system

$$H_{q+i,q+j,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, r-q; \end{matrix} \; k = 1, \ldots, q\right),$$

then there is no sub-group of order $(r-q)$, all of whose operators are independent of $X_1, \ldots, X_q$; that is, if there is a sub-group of order $(r-q)$ at all it must have at least one of its operators dependent on $X_1, \ldots, X_q$. In this case we should take, in order to form the functions $H_{q+i,q+j,k}$, some other set of $(r-q)$ operators out of the set $X_1, \ldots, X_r$ in place of $X_{q+1}, \ldots, X_r$; for there is no sub-group of order $(r-q)$ which cannot be expressed in some one of these ways.

We see this more clearly if we consider the sub-group $Y_{q+1}, \ldots, Y_r$ where

$$Y_{q+t} \equiv \sum^{k=r} a_{q+t,k} X_k, \qquad (t = 1, \ldots, r-q),$$

$a_{q+t,k}, \ldots$ being a set of constants.

This sub-group could then only fail to be expressible in the form

$$Y_{q+t} \equiv X_{q+t} - \sum^{k=q} h_{q+t,k} X_k, \qquad (t = 1, \ldots, r-q),$$

when

$$\begin{vmatrix} a_{q+1,q+1}, & \cdot & \cdot & a_{q+1,r} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{r,q+1}, & \cdot & \cdot & a_{r,r} \end{vmatrix} = 0;$$

and it could only fail to be expressible in some one of the required forms if all $(r-q)$-rowed determinants of the matrix

$$\begin{Vmatrix} a_{q+1,1}, & \cdot & \cdot & \cdot & a_{q+1,r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r,1}, & \cdot & \cdot & \cdot & a_{r,r} \end{Vmatrix}$$

vanished; that is, if the sub-group was of order less than $(r-q)$.

If on the other hand we find a set of values of $h_{q+t,\mu}, \ldots$ to satisfy the equations

$$H_{q+i,q+j,k} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, r-q; \end{pmatrix} \quad k = 1, \ldots, q \Big),$$

then $H_{q+i,q+j,q+t}, \ldots$ will be the structure constants of the sub-group whose operators are

$$X_{q+t} - \sum^{\mu=q} h_{q+t,\mu} X_\mu, \qquad (t = 1, \ldots, r-q).$$

We then denote the operators of this sub-group by $Y_{q+1}, \ldots, Y_r$ and the group itself by $Y_1, \ldots, Y_r$.

The sub-group is of course invariant for the operators $Y_{q+1}, \ldots, Y_r$; it will be invariant for

$$e_1 Y_1 + \ldots + e_q Y_q$$

if $\quad e_1 H_{1,q+i,k} + \ldots + e_q H_{q,q+i,k} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r-q \\ k = 1, \ldots, \quad q \end{pmatrix}.$

We therefore, in order to find within what group $Y_{q+1}, \ldots, Y_r$ is invariant, write down the matrix

$$\begin{Vmatrix} H_{1,q+i,k}, & \cdot & \cdot & \cdot \\ H_{2,q+i,k}, & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ H_{q,q+i,k}, & \cdot & \cdot & \cdot \end{Vmatrix},$$

where in any row i takes all values from 1 to $(r-q)$, and k all values from 1 to q.

Suppose that the values of $h_{q+t,\mu}, \ldots$ now found are such as when substituted in this matrix will make all $(q-m+1)$-rowed determinants but not all $(q-m)$-rowed determinants of the matrix vanish, then the sub-group $Y_{q+1}, \ldots, Y_r$ is invariant for m operators independent of one another and of $Y_{q+1}, \ldots, Y_r$. The sub-group is therefore invariant within a group of order $r-q+m$, and there are only $(q-m)$ independent operators for which it is not invariant. We say, then, that the sub-group $Y_{q+1}, \ldots, Y_r$ is of *index* $(q-m)$.

§ 146. We now wish to find the sub-groups conjugate to $Y_{q+1}, \ldots, Y_r$, so we must consider what this sub-group is transformed into when we apply the infinitesimal transformation

$$(1) \qquad x_i' = x_i + t Y_j x_i, \qquad (i = 1, \ldots, n).$$

If $j > q$ the operators $Y_{q+1}, \ldots, Y_r$ will be transformed into operators dependent on $Y_{q+1}, \ldots, Y_r$; we need therefore only consider the case where $j \ngtr q$.

We saw in § 76 that, X'_k denoting the operator derived from X_k by replacing x_i by x'_i,

$$X'_k = X_k + t\sum^{\mu=r} c_{jk\mu} X_\mu.$$

Hence we now have, since $Y_1, \ldots, Y_r$ are operators with the structure constants $H_{jk\mu}, \ldots$,

$$(2) \qquad Y'_k = Y_k + t\sum^{\mu=r} H_{jk\mu} Y_\mu.$$

Now $Y'_{q+1}, \ldots, Y'_r$ are the operators of the sub-group conjugate to $Y_{q+1}, \ldots, Y_r$ obtained by applying the transformation (1); and therefore, since this is a sub-group of order $(r-q)$, and differs infinitesimally from $Y_{q+1}, \ldots, Y_r$, it cannot have operators dependent on $X_1, \ldots, X_q$. We may therefore take its operators to be

$$X_{q+1} - \sum^{\mu=q} h'_{q+1,\mu} X_\mu, \ldots, X_r - \sum^{\mu=q} h'_{r\mu} X_\mu,$$

where $h'_{q+j,\mu} = h_{q+j,\mu} - t\lambda_{q+j,\mu}$, and $\lambda_{q+j,\mu}, \ldots$ are functions of the variable parameters $h_{q+i,\mu}, \ldots$ whose forms must now be determined.

The operators $Y'_{q+1}, \ldots, Y'_r$ are operators of the sub-group

$$X_{q+1} - \sum^{\mu=q} h'_{q+1,\mu} X_\mu, \ldots, X_r - \sum^{\mu=q} h'_{r\mu} X_r;$$

that is, of the sub-group

$$Y_{q+1} + t\sum^{\mu=q} \lambda_{q+1,\mu} Y_\mu, \ldots, Y_r + \sum^{\mu=q} \lambda_{r\mu} Y_\mu;$$

and therefore

$$Y'_{q+i} = \sum^{s=r-q} e_{q+i,q+s}\left(Y_{q+s} + t\sum^{\mu=q} \lambda_{q+s,\mu} Y_\mu\right), \qquad (i = 1, \ldots, r-q),$$

where $e_{q+i,q+s}, \ldots$ are constants.

If we now compare this expression for Y'_{q+j} with the expression obtained in (2), and equate the coefficients of $Y_{q+1}, \ldots, Y_r$ we see that, neglecting small quantities of the order t, $e_{q+i,q+s}$ is equal to unity or zero according as i is or is not equal to s; and therefore we see that

$$\lambda_{q+s,\mu} = H_{j,q+s,\mu}, \qquad \begin{pmatrix} s = 1, \ldots, r-q \\ \mu = 1, \ldots, \quad q \end{pmatrix}.$$

Since $j \ngtr q$, $H_{j,q+s,\mu} = \Pi_{j,q+s,\mu}$; and therefore the constants $h'_{q+j,\mu}, \ldots$ which define the sub-group conjugate to $Y_{q+1}, \ldots, Y_r$ obtained by the infinitesimal transformation

$$x'_i = x_i + tY_j x_i, \qquad (i = 1, \ldots, n)$$

are given by

$$h'_{q+i,\mu} = h_{q+i,\mu} - t\Pi_{j,\,q+i,\mu}, \qquad \begin{pmatrix} i = 1, \ldots, r-q \\ \mu = 1, \ldots, \quad q \end{pmatrix}.$$

Because the sub-group is invariant for the transformations

$$x'_i = x_i + tY_{q+j}\, x_i, \qquad \begin{pmatrix} i = 1, \ldots, n \\ j = 1, \ldots, r-q \end{pmatrix},$$

we see that for such transformations

$$h'_{q+i,\mu} = h_{q+i,\mu}.$$

We now want to find the constants defining the sub-group adjacent to that defined by $h_{q+i,\mu}, \ldots$ and obtained by the infinitesimal transformation

$$x'_i = x_i + (e_1 X_1 + \ldots + e_r X_r)\, x_i, \qquad (i = 1, \ldots, n).$$

We have

$$e_1 X_1 + \ldots + e_r X_r = \sum^{\mu=q} (e_\mu + \sum^{j=r-q} e_{q+j}\, h_{q+j,\mu})\, X_\mu + \sum^{j=r-q} e_{q+j}\, Y_{q+j},$$

and therefore

$$h'_{q+i,\mu} = h_{q+i,\mu} - \sum^{k=q} (e_k + \sum e_{q+j}\, h_{q+j,k})\, \Pi_{k,\,q+i,\mu}.$$

Now, since

$$H_{q+i,q+j,\mu} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, r-q; \end{pmatrix} \mu = 1, \ldots, q\Big),$$

$$\Pi_{q+j,q+i,\mu} = \sum^{k=q} h_{q+j,k}\, \Pi_{k,q+i,\mu},$$

and therefore

$$h'_{q+i,\mu} = h_{q+i,\mu} - \sum^{k=r} e_k\, \Pi_{k,\,q+i,\mu},$$
$$= h_{q+i,\mu} - (e_1 \Pi_1 + \ldots + e_r \Pi_r)\, h_{q+i,\mu}.$$

The relation between the groups $\Pi_1, \ldots, \Pi_r$ and $X_1, \ldots, X_r$ can now be expressed in general terms. Let $h_{q+t,\mu}, \ldots$ be a set of constants defining a sub-group of $X_1, \ldots, X_r$; then the set of constants $h'_{q+t,\mu}, \ldots$ which define the sub-group conjugate to this and obtained by the transformation

$$x'_i = e^{e_1 X_1 + \ldots + e_r X_r}\, x_i, \qquad (i = 1, \ldots, n)$$

are given by the formulae

$$h'_{q+t,\mu} = e^{-e_1\Pi_1-\ldots-e_r\Pi_r}\, h_{q+t,\mu}, \qquad \begin{pmatrix} t = 1, \ldots, r-q \\ \mu = 1, \ldots, \quad q \end{pmatrix}.$$

§ 147. In order to find all types of sub-groups of order $(r-q)$ we therefore proceed as follows.

If no sets of values of $h_{q+t,\mu}, \ldots$ can be obtained to satisfy the equations

$$(1) \quad H_{q+i,q+j,k} = 0, \qquad \begin{pmatrix} i = 1, \ldots, r-q\,; \\ j = 1, \ldots, r-q\,; \end{pmatrix} k = 1, \ldots, q \Big),$$

no sub-group of order $(r-q)$ exists.

If on the other hand such a set exists, let $h^0_{q+t,\mu}, \ldots$ satisfy the equations (1); we write down the matrix of the operators $\Pi_1, \ldots, \Pi_r$

$$\left\| \begin{matrix} \Pi_{1,q+j,k}, & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Pi_{r,q+j,k}, & \cdot & \cdot & \cdot \end{matrix} \right\|,$$

where in any row all values of j from 1 to $(r-q)$ and all values of k from 1 to q are to be taken. If when we substitute for $h_{q+t,\mu}, \ldots$ in this matrix the respective values $h^0_{q+t,\mu}, \ldots$ all $(s+1)$-rowed determinants of the matrix, but not all s-rowed determinants, vanish, then the sub-group is of index s; and the 'point' whose coordinates are $h^0_{q+t,\mu}, \ldots$ is of order s with respect to the equation system

$$(1) \quad H_{q+i,q+j,k} = 0 \text{ (in the variables } h_{q+t,\mu}, \ldots)$$

admitting the operators $\Pi_1, \ldots, \Pi_r$.

Since

$$(2) \quad H_{q+i,q+j,k} = \Pi_{q+i,q+j,k} - \sum^{\mu=q} h_{q+i,\mu}\, \Pi_{\mu,q+j,k},$$

the index s cannot exceed q.

We now find (as explained in § 103) the contracted operators of $\Pi_1, \ldots, \Pi_r$ with respect to the equation system which consists of (1) and the equations which define points of order s; for both of these equation systems are invariant under the operations of the group $\Pi_1, \ldots, \Pi_r$.

Let this combined equation system be

$$(3) \quad h_{q+t,\mu} = \phi_{q+t,\mu}(h_1, \ldots, h_p), \qquad \begin{pmatrix} t = 1, \ldots, r-q \\ \mu = 1, \ldots, \quad q \end{pmatrix},$$

where $h_1, \ldots, h_p$ are some unconnected parameters, in terms

of which those values of $h_{q+t,\mu}, \ldots$ can be expressed which satisfy the combined equations; and let $P_1, \ldots, P_r$ be the contracted operators.

Since $X_1, \ldots, X_r$ is isomorphic with $\Pi_1, \ldots, \Pi_r$ and $\Pi_1, \ldots, \Pi_r$ is isomorphic with $P_1, \ldots, P_r$, $X_1, \ldots, X_r$ must be isomorphic with $P_1, \ldots, P_r$; but the isomorphism is simple, only when $P_1, \ldots, P_r$ are independent operators.

Since the parameters of a sub-group of order $(r-q)$ and index s are by (3) expressible in terms of $h_1, \ldots, h_p$ we call these parameters the coordinates of the sub-group. From the definition of a point of order s exactly s of the operators $P_1, \ldots, P_r$ will be unconnected; and as these are operators in the variables $h_1, \ldots, h_p$ we conclude that $p \nless s$, and that there will be $(p-s)$ invariants, which we may take to be

$$h_{s+1}, \ldots, h_p.$$

If then $h_1, \ldots, h_p$ are the coordinates of a sub-group of index s and order $(r-q)$, the coordinates of the sub-group conjugate to this obtained by the transformation

$$x_i' = e^{e_1 X_1 + \ldots + e_r X_r} x_i, \qquad (i = 1, \ldots, n)$$

are given by

$$h_i' = e^{e_1 P_1 + \ldots + e_r P_r} h_i, \qquad (i = 1, \ldots, p).$$

Since s of the operators of the group $P_1, \ldots, P_r$ are unconnected, we can pass, by the operations of this group, from any point whose coordinates are $h_1^0, \ldots, h^0$, to any point whose coordinates are $h_1, \ldots, h_s, h_{s+1}^0, \ldots, h_p^0$. Sub-groups of the same order are therefore divided into classes according to their indices; only sub-groups of the same order and index can be conjugate; and of sub-groups of the same order and index only those can be conjugate for which the coordinates $h_{s+1}, \ldots, h_p$ are the same. There are therefore ∞^{p-s} different types of sub-groups of order $(r-q)$ and index s; and corresponding to any one of these types we have ∞^s conjugate sub-groups.

§ 148. We can apply these results to obtain the stationary functions of groups whose structure constants are assigned; and thus complete the investigation of which Chapters V and XII formed a part, viz. the determination of all possible types of groups with assigned structure constants.

Suppose the group $X_1, \ldots, X_r$ is in standard form so that $x_1, \ldots, x_m, x_{m+q+1}, \ldots, x_n$ are the invariants, and the stationary

functions only involve $x_1, \ldots, x_s$. If $x_1^0, \ldots, x_n^0$ is a point of general position then the group of the point—that is, the sub-group of operations leaving the point at rest—is of order $(r-q)$; and the coordinates of this group depend only on $x_1^0, \ldots, x_s^0$; for we have proved in § 112 that the equations

$$x_1 = x_1^0, \ldots, x_s = x_s^0$$

define the locus of points whose groups are the same as the group of $x_1^0, \ldots, x_n^0$.

Now by the operations of the group $X_1, \ldots, X_r$, only the coordinates $x_{m+1}, \ldots, x_{m+q}$ can vary; and, as there are $(r-s+m)$ independent infinitesimal transformations which leave $x_{m+1}^0, \ldots, x_s^0$ at rest, there will be $(r-s+m)$ infinitesimal transformations which do not transform the group of $x_1^0, \ldots, x_n^0$.

This group is therefore of index $(s-m)$; and its coordinates are expressible in terms of s parameters.

In order, therefore, to find the stationary functions of a group, when we are merely given the structure constants, we form the equations defining sub-groups of order $(r-q)$ and index $(s-m)$; the coordinates, then, of the sub-group which leaves a point of general position at rest will be expressible in terms of s parameters.

If the combined equation system is

$$h_{q+t,\mu} = \phi_{q+t,\mu}(h_1, \ldots, h_s), \quad \begin{pmatrix} t = 1, \ldots, r-q \\ \mu = 1, \ldots, \quad q \end{pmatrix},$$

then the stationary functions $f_{q+t,\mu}(x_1^0, \ldots, x_n^0)$ will be given by

$$f_{q+t,\mu}(x_1^0, \ldots, x_n^0) = \phi_{q+t,\mu}(h_1, \ldots, h_s).$$

Since the functions $\phi_{q+t,\mu}(h_1, \ldots, h_s), \ldots$ cannot be expressed in terms of a smaller number of arguments, we may express $h_1, \ldots, h_s$ in terms of $x_1^0, \ldots, x_n^0$; and by a change of variables we may take $h_1, \ldots, h_s$ to be respectively $x_1^0, \ldots, x_s^0$.

As we can vary $x_1^0, \ldots, x_s^0$ in any way we like, we see that we may take the stationary functions to be

$$\phi_{q+t,\mu}(x_1, \ldots, x_s), \qquad \begin{pmatrix} t = 1, \ldots, r-q \\ \mu = 1, \ldots, \quad q \end{pmatrix}.$$

When we have thus found the stationary functions of the group $X_1, \ldots, X_r$ we may complete the determination of the operators by the method explained in Chapter XII; and if any group with the assigned structure constants, and the

assigned numbers s, m and n exists, we can find it by the method now explained.

Such a group may not exist; thus if we take $r > 3$, $n = 1$, $m = 0$ and $s = 1$, we may, for many assigned sets of structure constants, construct the functions $\phi_{q+t,\mu}, \ldots$ which express the coordinates of sub-groups of order $(n-1)$ in terms of one parameter; but the operators $X_1, \ldots, X_r$ in one variable, which we should hence deduce, would not be independent; for (as we shall prove later), no group whose order exceeds three can exist in one variable.

§ 149. *Example.* Find all the sub-groups of order 3 of the group whose structure is given by

$$(1)\quad (X_2, X_3) = X_1, \quad (X_3, X_1) = X_2, \quad (X_1, X_2) = X_3,$$
$$(X_1, X_4) = 0, \quad (X_2, X_4) = 0, \quad (X_3, X_4) = 0.$$

We first find the sub-groups which can be expressed in the form $\quad X_1 - \lambda_1 X_4, \quad X_2 - \lambda_2 X_4, \quad X_3 - \lambda_3 X_4,$

that is, the sub-groups not containing X_4 as an operator.

Since $\quad (X_2 - \lambda_2 X_4,\ X_3 - \lambda_3 X_4) = (X_2, X_3) = X_1,$

we cannot express this alternant in terms of the operators of the sub-group unless $\lambda_1 = 0$. Similarly we see that we must have $\lambda_2 = 0$, and $\lambda_3 = 0$.

There is, therefore, only one sub-group of this form, viz. the self-conjugate sub-group X_1, X_2, X_3.

Whenever by this method we find only a discrete number of solutions of the equation system

$$H_{q+i,q+j,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, r-q; \end{matrix} \quad k = 1, \ldots, q\right),$$

the sub-groups must be self-conjugate; for if they had conjugate sets obtained by the infinitesimal transformation

$$x_i' = x_i + (e_1 X_1 + \ldots + e_r X_r)\, x_i, \qquad (i = 1, \ldots, n),$$

there would be an infinity of sub-groups of the required class.

We next find all sub-groups of order 3 which do not contain X_1 as an operator.

The general method of forming equations for $h_{q+t,\mu}, \ldots$ to define sub-groups of order $(r-q)$ is simplified when $q = 1$. If we take $\quad X_2 - h_2 X_1, \ldots, X_r - h_r X_1$

to be the operators of the sub-group of order $(r-1)$, then the equations which $h_2, \ldots, h_r$ must satisfy are

$$H_{ij} = h_i H_{1j} - h_j H_{1i}, \qquad \begin{pmatrix} i = 1, \ldots, r \\ j = 1, \ldots, r \end{pmatrix},$$

where $$H_{ij} = c_{ij1} + c_{ij2} h_2 + \ldots + c_{ijr} h_r.$$

In the example before us

$$H_{2,3} = 1,\ H_{2,4} = 0,\ H_{3,4} = 0,\ H_{1,4} = 0,\ H_{1,2} = h_3,\ H_{1,3} = -h_2;$$

and the equations defining the sub-group are therefore

$$h_4 = 0, \quad 1 + h_2^2 + h_3^2 = 0.$$

The sub-group sought has therefore the operators

$$X_2 - i\cos\theta X_1, \quad X_3 - i\sin\theta X_1, \quad X_4,$$

where θ is a variable parameter and i is the symbol $\sqrt{-1}$. By varying θ we get an infinity of conjugate sub-groups; and as the sub-group is not self-conjugate it must be of index unity.

By interchanging X_1 and X_2 we should obtain the system of conjugate sub-groups

$$X_1 - i\cos\phi X_2, \quad X_3 - i\sin\phi X_2, \quad X_4,$$

these two systems coincide, however, the relation between the parameters being $\cos\theta\cos\phi + 1 = 0$.

By interchanging X_1 and X_3 we get

$$X_2 - i\cos\psi X_3, \quad X_1 - i\sin\psi X_3, \quad X_4,$$

which also coincides with the first system, the relation between the parameters being $\sin\theta\sin\psi + 1 = 0$.

If we try to find a group in the single variable x which shall have the structure (1) we must take

$$X_2 = \phi_2(x) X_1, \quad X_3 = \phi_3(x) X_1, \quad X_4 = \phi_4(x) X_1.$$

We now have the following identities which enable us to determine the stationary functions

$$\phi_2(x) = i\cos x, \quad \phi_3(x) = i\sin x, \quad \phi_4(x) = 0;$$

and we see that the operators cannot be independent, X_4 being identically zero.

Now we know that in general $X_1 \phi_{q+t,k} = \Pi_{1,q+t,k}$; and in this example

$$\Pi_{121} = c_{121} + c_{122} h_2 + c_{123} h_3 + c_{124} h_4 = \quad h_3 = \quad i\sin x,$$
$$\Pi_{131} = c_{131} + c_{132} h_2 + c_{133} h_3 + c_{134} h_4 = -h_2 = -i\cos x;$$

so that, from either of these two equations, we see that, if $X_1 = \xi_1 \frac{\partial}{\partial x}$, then $\xi_1 = -1$, and therefore $X_1 = -\frac{\partial}{\partial x}$, and the group is

$$X_1 = -\frac{\partial}{\partial x}, \quad X_2 = i \cos x \frac{\partial}{\partial x}, \quad X_3 = i \sin x \frac{\partial}{\partial x}, \quad X_4 = 0.$$

§ 150. *Example.* Find the sub-groups of order 2 and index 2 of the group

$$(X_2, X_3) = X_1, \quad (X_3, X_1) = X_2, \quad (X_1, X_2) = X_3,$$
$$(X_1, X_4) = 0, \quad (X_2, X_4) = 0, \quad (X_3, X_4) = 0.$$

We shall only find those which are of the form

$$X_3 - h_{31} X_1 - h_{32} X_2, \quad X_4 - h_{41} X_1 - h_{42} X_2.$$

Applying the rule (or otherwise) we find the conditions for a group are

$$h_{4,2}(1 + h_{3,1}^2) - h_{4,1} h_{3,2} h_{3,1} = 0,$$
$$h_{4,1}(1 + h_{3,2}^2) - h_{4,2} h_{3,1} h_{3,2} = 0,$$

so that $$1 + h_{3,1}^2 + h_{3,2}^2 = 0.$$

We must therefore take (λ and θ being parameters)

$$h_{3,1} = i \cos\theta, \quad h_{3,2} = i \sin\theta, \quad h_{4,1} = \lambda \sin\theta, \quad h_{4,2} = -\lambda \cos\theta;$$

and we may directly verify that

$$(X_3 - i\cos\theta X_1 - i \sin\theta X_2, \quad X_4 - \lambda \sin\theta X_1 + \lambda \cos\theta X_2)$$
$$= -i\lambda (X_3 - i \cos\theta X_1 - i \sin\theta X_2).$$

In order to find the corresponding group in the two variables x, y we suppose that

$$X_3 = \phi_{3,1} X_1 + \phi_{3,2} X_2, \quad X_4 = \phi_{4,1} X_1 + \phi_{4,2} X_2.$$

Since the index is 2 we have $s - m = 2$; and, since in general s cannot exceed n, in this example, s cannot exceed 2, so that $m = 0$ and $s = 2$; that is, the group is non-stationary. The order of the group of the point x^0, y^0 of general position is $(r-q)$, and therefore $(r-q) = 2$; and as $r = 4$ we must take $q = 2$, so that the group is transitive, and X_1 and X_2 must be unconnected.

We have

$$\phi_{3,1}(x, y) = i \cos\theta, \quad \phi_{4,1}(x, y) = \lambda \sin\theta,$$
$$\phi_{3,2}(x, y) = i \sin\theta, \quad \phi_{4,2}(x, y) = -\lambda \cos\theta.$$

We may then by a change of the variables take

$$\phi_{3,1}(x, y) = x, \quad \text{and} \quad \phi_{4,1}(x, y) = y,$$

and therefore

$$\phi_{3,2}(x, y) = i\,(1+x^2)^{\frac{1}{2}}, \quad \phi_{4,2}(x, y) = -iyx\,(1+x^2)^{-\frac{1}{2}}.$$

We have

$$\begin{aligned} X_1\phi_{3,1} &= \Pi_{131} = -h_{3,2}\,h_{3\,1} = -ix\,(1-x^2)^{\frac{1}{2}}, \\ X_1\phi_{4,1} &= \Pi_{141} = -h_{3,1}\,h_{4,2} = iyx^2\,(1+x^2)^{-\frac{1}{2}}, \\ X_2\phi_{3,1} &= \Pi_{231} = \quad 1+h_{3,1}^2 = 1+x^2, \\ X_2\phi_{4,1} &= \Pi_{241} = \quad h_{4,1}\,h_{3,1} = xy. \end{aligned}$$

We then see that

$$X_1 = -ix\,(1+x^2)^{\frac{1}{2}}\frac{\partial}{\partial x} + iyx^2\,(1+x^2)^{-\frac{1}{2}}\frac{\partial}{\partial y},$$

$$X_2 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$$

$$X_3 = i\,(1+x^2)^{\frac{1}{2}}\frac{\partial}{\partial x} - iyx\,(1+x^2)^{-\frac{1}{2}}\frac{\partial}{\partial y}.$$

Now X_4 is identically zero, and therefore there is no group of order 4 of the given structure, but X_1, X_2, X_3 will with $X_4 \equiv 0$ form a group of order 3 with the required structure.

§ 151. When the sub-group whose conjugate sub-groups are required is of order 1 the equations

$$H_{q+i,\,q+j,\,k} = 0, \qquad \left(\begin{matrix} i = 1, \ldots, r-q; \\ j = 1, \ldots, r-q; \end{matrix} \quad k = 1, \ldots, q\right)$$

are satisfied identically, since $q = r-1$.

The variables which define the sub-group are $h_{r1}, \ldots, h_{r\;r-1}$; and

$$e_1X_1 + \ldots + e_rX_r$$

will be the operator of this sub-group if

$$e_k + h_{rk}e_r = 0, \qquad (k = 1, \ldots, r-1).$$

We therefore take $h_{rk} = \dfrac{e_k}{e_r}$, and let

$$E_\mu = \sum^{k=j=r} c_{j\mu k}\, e_j \frac{\partial}{\partial e_k}.$$

In operating on any function of $h_{r1}, \ldots, h_{r,\,r-1}$

$$\frac{\partial}{\partial e_k} = \frac{\partial h_{r1}}{\partial e_k}\frac{\partial}{\partial h_{r1}} + \ldots + \frac{\partial h_{r,\,r-1}}{\partial e_k}\frac{\partial}{\partial h_{r,\,r-1}},$$

so that

$$e_j\frac{\partial}{\partial e_k} = h_{rj}\frac{\partial}{\partial h_{rk}} \text{ if } j < r \text{ and } k < r,$$

$$e_r\frac{\partial}{\partial e_k} = -\frac{\partial}{\partial h_{rk}} \text{ if } k < r,$$

$$e_j\frac{\partial}{\partial e_r} = \sum^{k=r-1} h_{rj}\,h_{rk}\frac{\partial}{\partial h_{rk}} \text{ if } j < r,$$

$$e_r\frac{\partial}{\partial e_r} = \sum^{k=r-1} h_{rk}\frac{\partial}{\partial h_{rk}}.$$

Therefore, since

$$E_\mu = \sum^{j=k=r-1} c_{j\mu k}\,e_j\frac{\partial}{\partial e_k} + \sum^{k=r-1} c_{r\mu k}\,e_r\frac{\partial}{\partial e_k} + \sum^{j=r-1} c_{j\mu r}\,e_j\frac{\partial}{\partial e_r} + c_{r\mu r}\,e_r\frac{\partial}{\partial e_r}$$

$$= \sum^{k=r-1}\left(\sum^{j=r-1} c_{j\mu k}h_{rj} + c_{\mu rk} + c_{\mu rr}\,h_{rk} + \sum^{j=r-1} c_{j\mu r}\,h_{rj}\,h_{rk}\right)\frac{\partial}{\partial h_{rk}}$$

$$= \sum^{k=r-1}\Pi_{\mu rk}\frac{\partial}{\partial h_{rk}} \text{ if } \mu < r,$$

we see that in operating on any function of $h_{r1}, \ldots, h_{r,\,r-1}$ E_μ has the same effect as Π_μ if $\mu < r$.

Since $$e_1E_1 + \ldots + e_rE_r \equiv 0,$$

$$E_r = -\frac{e_1}{e_r}E_1 - \ldots - \frac{e_{r-1}}{e_r}E_{r-1}$$

$$= h_{r1}\,\Pi_1 + \ldots + h_{r,\,r-1}\,\Pi_{r-1},$$

and this operator is equivalent to Π_r, since the equations

$$H_{q+i,\,q+j,\,k} = 0$$

are satisfied for all values of $h_{r1}, \ldots, h_{r,\,r-1}$.

Since the coordinates of the sub-group of order one are the ratios of $e_1, \ldots, e_r$, we see that for such sub-groups the operators $\Pi_1, \ldots, \Pi_r$ may be replaced by the known operators $E_1, \ldots, E_r$, of which we made use in Chapter V.

CHAPTER XIV

ON PFAFF'S EQUATION AND THE INTEGRALS OF PARTIAL DIFFERENTIAL EQUATIONS

§ 152. If $x_1, \ldots, x_n$ are the coordinates of a point in n-way space, and

$$(x'_1 - x_1) p_1 + \ldots + (x'_n - x_n) p_n = 0$$

(where $x'_1, \ldots, x'_n$ are the current coordinates) the equation of a plane through $x_1, \ldots, x_n$, then we speak of the point together with the plane as an *element* of this space. We say that the coordinates of the element are $x_1, \ldots, x_n, p_1, \ldots, p_n$, where $x_1, \ldots, x_n$ are the coordinates of the point of the element, and $p_1, \ldots, p_n$ the coordinates of the plane of the element. In the coordinates of the plane we are only concerned with the ratios $p_1 : p_2 \ldots : p_n$; and therefore in n-way space there are ∞^{2n-1} elements.

Two contiguous elements, $x_1, \ldots, x_n, p_1, \ldots, p_n$ and

$$x_1 + dx_1, \ldots, x_n + dx_n, \;\; p_1 + dp_1, \ldots, p_n + dp_n,$$

are said to be *united* if the point of one element lies on the plane of the other. More exactly expressed, the elements are united if the point of the second is distant from the plane of the first by a small quantity of the second order. The analytical condition for this is

$$(1) \qquad p_1 dx_1 + \ldots + p_n dx_n = 0;$$

and therefore, if this equation is satisfied, the point of the first element is also distant from the plane of the second element by a small quantity of the second order.

The equation (1) is called Pfaff's equation.

Since the coordinates of an element only involve $p_1, \ldots, p_n$ through their ratios, we shall suppose that, when we are given any equation connecting the coordinates

$$x_1, \ldots, x_n, p_1, \ldots, p_n$$

of an element, it is one which is homogeneous in $p_1, \ldots, p_n$.

If we have m unconnected equations connecting

$$x_1, \ldots, x_n, p_1, \ldots, p_n,$$

viz.

$$(2) \quad f_i(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0, \qquad (i = 1, \ldots, m)$$

then ∞^{2n-m-1} elements of space will satisfy this equation system; they will be called the elements of the system.

Two contiguous elements of the system will not however, in general, be united. The question thus arises, what are the necessary and sufficient conditions which these equations must satisfy in order that any two contiguous elements of the system may be united? In other words, what are the conditions that the equations (2) may satisfy Pfaff's equation?

Suppose if possible that, from the equations (2), no equation of the form

$$f(x_1, \ldots, x_n) = 0$$

can be deduced; we must then be able to express m of the coordinates $p_1, \ldots, p_n$ in terms of the remaining coordinates of the element $x_1, \ldots, x_n, p_1, \ldots, p_n$. The equation system may therefore be thrown into the form

$$(3) \quad p_1 = f_1(x_1, \ldots, x_n, p_{m+1}, \ldots, p_n), \ldots,$$
$$p_m = f_m(x_1, \ldots, x_n, p_{m+1}, \ldots, p_n),$$

or into some equivalent form, obtained by replacing the suffixes $1, \ldots, m$ by some m of the suffixes $1, \ldots, n$. It is obvious that, by differentiating the equations (3), we could not obtain any equation connecting $dx_1, \ldots, dx_n$, and could not therefore by the equation system assumed satisfy Pfaff's equation.

We must therefore suppose that the equation system (2) is such that at least one equation between $x_1, \ldots, x_n$ alone can be deduced from it. Suppose that exactly s of these equations can be deduced; and suppose further that these have been thrown into the forms

$$x_n = f_n(x_1, \ldots, x_{n-s}), \ldots, x_{n-s+1} = f_{n-s+1}(x_1, \ldots, x_{n-s}).$$

We now have

$$p_1 dx_1 + \ldots + p_n dx_n = \sum_{i=1}^{i=n-s} \Big(p_i + \sum_{t=1}^{t=s} p_{n-s+t} \frac{\partial f_{n-s+t}}{\partial x_i}\Big) dx_i;$$

and therefore, if the equations (2) are to satisfy Pfaff's equation, we must have

$$p_i + \sum_{t=1}^{t=s} p_{n-s+t} \frac{\partial f_{n-s+t}}{\partial x_i} = 0, \qquad (i = 1, \ldots, n-s);$$

for, by hypothesis, $x_1, \ldots, x_{n-s}$ are unconnected.

We therefore conclude that every equation system satisfying Pfaff's equation must include the system

$$x_n = f_n(x_1, \dots, x_{n-s}), \dots, x_{n-s+1} = f_{n-s+1}(x_1, \dots, x_{n-s}),$$

$$p_{m-s} + \sum_{t=s}^{t=s} p_{n-s+t} \frac{\partial f_{n-s+t}}{\partial x_{n-s}} = 0, \dots, p_1 + \sum^{t=s} p_{n-s+t} \frac{\partial f_{n-s+t}}{\partial x_1} = 0.$$

To these equations we may add a number of arbitrary equations connecting $x_1, \dots, x_{n-s}, p_{n-s+1}, \dots, p_n$; these equations, however, must be such that no equation of the form

$$f(x_1, \dots, x_{n-s}) = 0$$

is deducible from them.

A set of equations satisfying Pfaff's equation is called a *Pfaffian system*. If the system contains m unconnected equations it is said to be of *order* m, and we have proved that $m \not< n$. When the number m is not specified it is to be understood as being equal to n, and a Pfaffian system as being of order n unless expressly stated to be of order m.

The equations of the system which do not involve $p_1, \dots, p_n$ will be called the *generating* equations. There must be at least one generating equation, and there cannot be more than n; there are, therefore, n classes of generating equations, if we measure the class by the number of unconnected generating equations in the system.

§ 153. We now proceed to express in a convenient form the conditions that n equations should form a Pfaffian system.

Let v be *any* function of the variables $x_1, \dots, x_n, p_1, \dots, p_n$; and let $\bar{v}$ denote the operator

$$\frac{\partial v}{\partial p_1}\frac{\partial}{\partial x_1} + \dots + \frac{\partial v}{\partial p_n}\frac{\partial}{\partial x_n} - \frac{\partial v}{\partial x_1}\frac{\partial}{\partial p_1} - \dots - \frac{\partial v}{\partial x_n}\frac{\partial}{\partial p_n};$$

then, u being any function of the variables,

$$\bar{v} . u = \sum^{i=n} \left(\frac{\partial v}{\partial p_i}\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial p_i} \right).$$

We call the expression on the right the *alternant* of the functions v and u, and we denote it by (v, u); we have

$$\bar{v} . u = (v, u) = -(u, v) = -\bar{u} . v.$$

The equation $v = 0$ will admit the infinitesimal transformation

$$x_i' = x_i + t\frac{\partial u}{\partial p_i}, \quad p_i' = p_i - t\frac{\partial u}{\partial x_i}, \qquad (i = 1, \dots, n)$$

if, and only if, the equation $(u, v) = 0$ is *connected* with $v = 0$; that is, if the values of the variables, which satisfy the second equation, also satisfy the first.

A set of functions $u_1, \ldots, u_m$ is said to be in *involution* when the alternant of every pair vanishes.

So also a set of equations,

$$u_1 = 0, \ldots, u_m = 0,$$

is said to be in *involution* when for all values of the suffixes $(u_i, u_j) = 0$ is an equation connected with the given set.

An equation system in involution,

$$(1) \qquad u_1 = 0, \ldots, u_m = 0,$$

will therefore admit the m infinitesimal transformations

$$(2) \quad x_i' = x_i + t\frac{\partial u_j}{\partial p_i}, \quad p_i' = p_i - t\frac{\partial u_j}{\partial x_i}, \qquad \begin{pmatrix} i = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix}.$$

If $v_1 = 0, \ldots, v_m = 0$ is any given equation system such that each of these equations is connected with

$$u_1 = 0, \ldots, u_m = 0,$$

and each of the equations $u_1 = 0, \ldots, u_m = 0$ is connected with $v_1 = 0, \ldots, v_m = 0$, we say that the two systems are *equivalent*.

We must now prove that, if any equation system is in involution, then any equivalent system is also in involution.

If $v = 0$ is connected with the system (1), it must admit all the infinitesimal transformations which (1) admits; and therefore

$$(v, u_1) = 0, \ldots, (v, u_m) = 0$$

are equations each of which is connected with (1).

If then $v_1 = 0, \ldots, v_m = 0$ is equivalent to (1) we know that the equation $(v_i, u_j) = 0$ will be connected with

$$u_1 = 0, \ldots, u_m = 0;$$

and therefore $u_1 = 0, \ldots, u_m = 0$ will admit the m infinitesimal transformations

$$(3) \quad x_i' = x_i + t\frac{\partial v_j}{\partial p_i}, \quad p_i' = p_i - t\frac{\partial v_j}{\partial k_i}, \qquad \begin{pmatrix} i = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix}.$$

Now each of the equations $v_1 = 0, \ldots, v_m = 0$ is connected with $u_1 = 0, \ldots, u_m = 0$; and therefore each of these equations admits the infinitesimal transformations (3); that is, the equation $(v_i, v_j) = 0$ is connected with $u_1 = 0, \ldots, u_m = 0$, and

therefore with $v_1 = 0, \dots, v_m = 0$; that is, $v_1 = 0, \dots, v_m = 0$ are equations in involution.

If $v_1, \dots, v_m$ is a set of functions of $x_1, \dots, x_n, p_1, \dots, p_n$, in terms of which we can express $u_1, \dots, u_m$; then, if $u_1, \dots, u_m$ are unconnected, we can express $v_1, \dots, v_m$ in terms of $u_1, \dots, u_m$; we say that two such systems of functions are *equivalent*.

When we say that a function is homogeneous we shall mean that it is homogeneous in $p_1, \dots, p_n$; suppose that $u_1, \dots, u_m$ are each homogeneous functions, then, if $v_1, \dots, v_m$ is an equivalent function system, v_j will not in general be a homogeneous function; but, since there are m homogeneous functions, equivalent to $v_1, \dots, v_m$, we shall say that $v_1, \dots, v_m$ is *a homogeneous function system.* When each of the functions $v_1, \dots, v_m$ is separately homogeneous, we shall say that the homogeneous function system is in *standard form.*

Similarly, if we say that the equation system

$$v_1 = 0, \dots, v_m = 0$$

is homogeneous, that will not mean that each separate equation is homogeneous, but only that an equivalent system can be found, viz.

$$u_1 = 0, \dots, u_m = 0,$$

each equation of which is homogeneous in $p_1, \dots, p_n$.

It can be at once verified that the n unconnected equations

$$x_n - f_n(x_1, \dots, x_{n-s}) = 0, \dots, x_{n-s+1} - f_{n-s+1}(x_1, \dots, x_{n-s}) = 0,$$

$$p_{n-s} + \sum^{t=s} p_{n-s+t}\frac{\partial f_{n-s+t}}{\partial x_{n-s}} = 0, \dots, p_1 + \sum^{t=s} p_{n-s+t}\frac{\partial f_{n-s+t}}{\partial x_1} = 0$$

are in involution; and that each of these equations is homogeneous; we have, therefore, the following theorem: *if m equations form a Pfaffian system, it is possible to deduce from them n unconnected homogeneous equations in involution.*

The most important Pfaffian systems are those in which $m = n$, and we see that *n equations cannot form a Pfaffian system unless they form a homogeneous equation system which is in involution.*

§ 154. We shall now prove the converse of this theorem, viz. that *a homogeneous equation system of order n in involution forms a Pfaffian system.*

Suppose that in the system there are s generating equations, viz. $f_1(x_1, \ldots, x_n) = 0, \ldots, f_s(x_1, \ldots, x_n) = 0$; and let the remaining $(n-s)$ equations of the system be thrown into the form

$$p_{s+1} - \phi_{s+1}(p_1, \ldots, p_s, x_1, \ldots, x_n) = 0, \ldots,$$
$$p_n - \phi_n(p_1, \ldots, p_s, x_1, \ldots, x_n) = 0,$$

where the functions $\phi_{s+1}, \ldots, \phi_n$ are homogeneous and of the first degree.

We must first prove that the Jacobian determinantal equation

$$(1) \qquad \begin{vmatrix} \frac{\partial f_1}{\partial x_1}, & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_s} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_s}{\partial x_1}, & \cdot & \cdot & \cdot & \frac{\partial f_s}{\partial x_s} \end{vmatrix} = 0$$

is not connected with the generating equations $f_1 = 0, \ldots, f_s = 0$.

Since $(p_{s+1} - \phi_{s+1}, f_1) = 0, \ldots, (p_{s+1} - \phi_{s+1}, f_s) = 0$,

$$\frac{\partial f_1}{\partial x_{s+1}} = \frac{\partial f_1}{\partial x_1}\frac{\partial \phi_{s+1}}{\partial p_1} + \ldots + \frac{\partial f_1}{\partial x_s}\frac{\partial \phi_{s+1}}{\partial p_s},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\frac{\partial f_s}{\partial x_{s+1}} = \frac{\partial f_s}{\partial x_1}\frac{\partial \phi_{s+1}}{\partial p_1} + \ldots + \frac{\partial f_s}{\partial x_s}\frac{\partial \phi_{s+1}}{\partial p_s};$$

and therefore, if the equation (1) were satisfied for those values of the variables which satisfy the generating equations, all s-rowed determinants of the matrix

$$\begin{Vmatrix} \frac{\partial f_1}{\partial x_1}, & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial x_{s+1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_s}{\partial x_1}, & \cdot & \cdot & \cdot & \frac{\partial f_s}{\partial x_{s+1}} \end{Vmatrix}$$

would, when equated to zero, be equations connected with the generating equations.

Proceeding thus, from the equations

$$(p_{s+2} - \phi_{s+2}, f_i) = 0, \ldots, (p_n - \phi_n, f_i) = 0, \qquad (i = 1, \ldots, s),$$

we should similarly see that all s-rowed determinants of the matrix

$$\begin{Vmatrix} \frac{\partial f_1}{\partial x_1}, & . & . & . & \frac{\partial f_1}{\partial x_n} \\ . & . & . & . & . \\ \frac{\partial f_s}{\partial x_1}, & . & . & . & \frac{\partial f_s}{\partial x_n} \end{Vmatrix}$$

would also, when equated to zero, be connected with the generating equations.

Now this is impossible; for, were it true, it would mean that, $x_1, \ldots, x_n$ being the coordinates of a point P on the $(n-s)$-way locus

$$f_1 = 0, \ldots, f_s = 0,$$

and $x_1 + dx_1, \ldots, x_n + dx_n$ the coordinates of a consecutive point P' on the $(n-s+1)$-way locus

$$f_1 = 0, \ldots, f_{s-1} = 0,$$

P' must also be on the $(n-s)$-way locus; and this is of course not true, since the equations which define the locus are unconnected.

The Jacobian determinantal equation is therefore unconnected with the generating equations; and we may therefore throw the equations of the given homogeneous involution system into the forms

$$x_1 - f_1(x_{s+1}, \ldots, x_n) = 0, \ldots, x_s - f_s(x_{s+1}, \ldots, x_n) = 0,$$
$$p_{s+1} - f_{s+1}(p_1, \ldots, p_s, x_{s+1}, \ldots, x_n) = 0, \ldots,$$
$$p_n - f_n(p_1, \ldots, p_s, x_{s+1}, \ldots, x_n) = 0,$$

where $f_{s+1}, \ldots, f_n$ are homogeneous of the first degree in $p_1, \ldots, p_s$.

By reason of the homogeneity of these functions we have

$$f_{s+j} = \sum^{i=s} p_i \frac{\partial f_{s+j}}{\partial p_i}, \qquad (j = 1, \ldots, n-s),$$

and, since $(p_{s+j} - f_{s+j}, x_i - f_i) = 0$, we have

$$\frac{\partial f_{s+j}}{\partial p_i} + \frac{\partial f_i}{\partial x_{s+j}} = 0;$$

we therefore conclude that

$$p_{s+j} + \sum^{i=s} p_i \frac{\partial f_i}{\partial x_{s+j}} = 0, \qquad (j = 1, \ldots, n-s).$$

From these $(n-s)$ equations together with

$$x_1-f_1=0,\ldots,x_s-f_s=0,$$

we now at once deduce Pfaff's equation.

We have therefore proved that *the necessary and sufficient conditions that n unconnected equations should form a Pfaffian system are that the equations should be homogeneous, and in involution.*

§ 155. We now know that ∞^{n-1} elements of space will satisfy any assigned Pfaffian system of n equations between the coordinates of the elements $x_1,\ldots x_n$, $p_1,\ldots p_n$. If the system contains only one generating equation, then the elements consist of the points of an $(n-1)$-way locus in this space together with the corresponding tangent planes to the locus. If there are two generating equations $f_1(x_1,\ldots,x_n)=0$, $f_2(x_1,\ldots,x_n)=0$ the elements consist of the points of this $(n-2)$-way locus together with the tangent planes which can be drawn at each point of the locus; there is not now, however, one definite plane at each point $x_1,\ldots,x_n$, but an infinity of tangent planes, viz.

$$(x_1'-x_1)\left(\lambda\frac{\partial f_1}{\partial x_1}+\mu\frac{\partial f_2}{\partial x_1}\right)+\ldots+(x_n'-x_n)\left(\lambda\frac{\partial f_1}{\partial x_n}+\mu\frac{\partial f_2}{\partial x_n}\right)=0,$$

where $\lambda:\mu$ is a variable parameter and $x_1',\ldots,x_n'$ are the current coordinates.

If there are three generating equations $f_1=0$, $f_2=0$, $f_3=0$ the elements will be formed by the points of this $(n-3)$-way locus together with the ∞^2 of tangent planes, viz.

$$(x_1'-x_1)\left(\lambda\frac{\partial f_1}{\partial x_1}+\mu\frac{\partial f_2}{\partial x_1}+\nu\frac{\partial f_3}{\partial x_1}\right)+\ldots$$
$$+(x_n'-x_n)\left(\lambda\frac{\partial f_1}{\partial x_n}+\mu\frac{\partial f_2}{\partial x_n}+\nu\frac{\partial f_3}{\partial x_n}\right)=0,$$

and so on.

Each of these different classes of ∞^{n-1} elements satisfying the Pfaffian equation

$$p_1\,dx_1+\ldots+p_n\,dx_n=0$$

will be denoted by the symbol M_{n-1}; each will form a manifold of united elements with $(n-1)$ 'degrees of freedom.'

Thus, when $n=2$, that is, in two-dimensional space, the elements are the points with the straight lines through the points. The symbol M_1 will now denote either an infinity of

points on some curve together with the corresponding tangents to the curve; or a fixed point with the infinity of straight lines through the point; either of these infinities of elements will satisfy the Pfaffian equation

$$p_1 dx_1 + p_2 dx_2 = 0.$$

In three-dimensional space there are ∞^5 elements consisting of points with the planes through them. The symbol M_2 will now denote one of three ∞^2 sets of united elements, viz. (1) the points of any surface with the corresponding tangent planes; (2) the infinity of points of any curve together with an infinity of tangent planes passing through each point of this curve; (3) the ∞^2 of planes passing through any fixed point; the elements of any one of these three sets will satisfy the Pfaffian equation

$$p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = 0.$$

§ 156. We must now consider Lie's definition of an integral of a partial differential equation of the first order; and we need only take the case where the equation is homogeneous, and the dependent variable does not explicitly occur; for any partial differential equation of the first order can be reduced to such a form (Forsyth, *Differential Equations*, § 209).

Let $$f(x_1, \ldots, x_n,\ p_1, \ldots, p_n) = 0$$

be such an equation; according to the usual definition $\phi(x_1, \ldots, x_n) = 0$ is said to be an integral if, and only if,

$$f\left(x_1, \ldots, x_n, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}\right) = 0 \text{ is connected with } \phi = 0.$$

Stated geometrically, any surface—that is, any $(n-1)$-way locus—is said to be an integral, if the coordinates of the tangent plane, at any point, are connected with the coordinates of the point by the equation

$$f(x_1, \ldots, x_n,\ p_1, \ldots, p_n) = 0.$$

Otherwise expressed, if we have any M_{n-1}, whose elements satisfy the given equation, and which has only one generating equation, then that generating equation is said to be an integral of the given equation. Lie extends the notion of an integral by defining it as the generating equations of *any* M_{n-1}, which includes, as one of its Pfaffian system, the given differential equation

$$f(x_1, \ldots, x_n,\ p_1, \ldots, p_n) = 0.$$

If then

$$f_1(x_1, \dots, x_n, p_1, \dots, p_n) = 0, \dots, f_n(x_1, \dots, x_n, p_1, \dots, p_n) = 0$$

is any homogeneous equation system in involution, such that $f = 0$ is connected with $f_1 = 0, \dots, f_n = 0$, the generating equations of this system will be an integral, whatever the number of these generating equations; whereas, according to the usual definition, they would only be an integral if the number was one. By this extension of the definition of an integral, it will be seen that more uniformity is introduced into the theory of the transformations of partial differential equations of the first order.

It should be noticed, however, that it is only special forms of differential equations which can admit these new integrals. If the equation

$$f(x_1, \dots, x_n, p_1, \dots, p_n) = 0$$

has an integral of the form

$$x_n = f_n(x_1, \dots, x_{n-s}), \dots, x_{n-s+1} = f_{n-s+1}(x_1, \dots, x_{n-s}),$$

the equation must be satisfied for all values of

$$x_1, \dots, x_{n-s}, p_{n-s+1}, \dots, p_n,$$

when we substitute in it for $x_n, \dots, x_{n-s+1}$ the respective functions $f_n, \dots, f_{n-s+1}$, and for p_k (where k may have any value from 1 to $(n-s)$), the sum

$$-\sum^{j=s} p_{n-s+j} \frac{\partial f_{n-s+j}}{\partial x_k}.$$

Now to satisfy these equations it would in general be necessary that the functions $f_n, \dots, f_{n-s+1}$ should satisfy a number of partial differential equations, and, this number being generally greater than s, the equations for $f_n, \dots, f_{n-s+1}$ would not usually be consistent.

If, however, the given differential equation is the linear one,

$$P_1 p_1 + \dots + P_n p_n = 0,$$

where $P_1, \dots, P_n$ are functions of $x_1, \dots, x_n$, it will admit these extended integrals. To prove this, let

$$u_1 = a_1, \dots, u_{n-1} = a_{n-1}$$

be the integral equations of any characteristic curve defined by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n};$$

then

$$P_1 \frac{\partial u_k}{\partial x_1} + \ldots + P_n \frac{\partial u_k}{\partial x_n} = 0, \qquad (k = 1, \ldots, n-1).$$

From these conditions it follows that

$$P_1 p_1 + \ldots + P_n p_n = 0,\ u_1 - a_1 = 0, \ldots,\ u_{n-1} - a_{n-1} = 0$$

are n homogeneous equations in involution; and therefore $u_1 - a_1 = 0, \ldots, u_{n-1} - a_{n-1} = 0$ are generating equations of a Pfaffian system, which includes the given linear equation; it follows that an integral of

$$P_1 p_1 + \ldots + P_n p_n = 0$$

will be $$u_1 = a_1, \ldots,\ u_{n-1} = a_{n-1},$$

where $a_1, \ldots, a_{n-1}$ are any constants.

§ 157. In order to find the complete integral of

$$f(x_1, \ldots, x_n,\ p_1, \ldots, p_n) = 0,$$

we must find $(n-1)$ other unconnected homogeneous equations, forming with $f = 0$ a Pfaffian system; the generating equations of this system will be (in Lie's sense) a complete integral if they involve $(n-1)$ effective arbitrary constants.

Suppose that

$$f_1(x_1, \ldots, x_n,\ p_1, \ldots, p_n) = 0, \ldots,\ f_m(x_1, \ldots, x_n,\ p_1, \ldots, p_n) = 0$$

are m given homogeneous equations in involution; we can throw these equations into such a form that some m of the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ will be given in terms of the remaining ones.

Let $x_1, \ldots, x_{m-s},\ p_1, \ldots, p_s$ be given by

$$x_i - f_i(x_{m-s+1}, \ldots, x_n,\ p_{s+1}, \ldots, p_n) = 0, \qquad (i = 1, \ldots, m-s)$$

$$p_j - \phi_j(x_{m-s+1}, \ldots, x_n,\ p_{s+1}, \ldots, p_n) = 0, \qquad (j = 1, \ldots, s).$$

These equations are still in involution; but in any such equation as $(x_i - f_i,\ p_j - \phi_j) = 0$ the variables $x_1, \ldots, x_{m-s}$, $p_1, \ldots, p_s$ do not occur at all; and it therefore follows that the above alternant, if it vanishes at all, must do so identically, and not by virtue of any equation system; the homogeneous function system

$$x_1 - f_1, \ldots, x_{m-s} - f_{m-s},\ p_1 - \phi_1, \ldots, p_s - \phi_s$$

must therefore be a system in involution.

If then we are given m equations in involution, and require the remaining $(n-m)$ equations forming with them a homogeneous Pfaffian system, we can reduce the problem to the following: given m homogeneous functions in involution, it is required to find $(n-m)$ other homogeneous functions, forming with the given functions a complete system in involution.

We shall show how one homogeneous function of degree zero may be obtained; having found this we shall have $(m+1)$ homogeneous functions in involution, and may proceed similarly till all the functions are obtained.

§ 158. Let $u_1, \dots, u_m$ be the given homogeneous functions in involution, then, $\bar{u}$ denoting the operator

$$\frac{\partial u}{\partial p_1}\frac{\partial}{\partial x_1}+\dots+\frac{\partial u}{\partial p_n}\frac{\partial}{\partial x_n}-\frac{\partial u}{\partial x_1}\frac{\partial}{\partial p_1}-\dots-\frac{\partial u}{\partial x_n}\frac{\partial}{\partial p_n},$$

we see that if v is any function of $u_1, \dots, u_m$

$$\bar{v}=\frac{\partial v}{\partial u_1}\bar{u}_1+\dots+\frac{\partial v}{\partial u_m}\bar{u}_m$$

(this result is of course true whether or not $u_1, \dots, u_m$ are in involution); the operator v is therefore connected with the operators $\bar{u}_1, \dots, \bar{u}_m$.

Conversely if $\bar{v}$ is connected with $\bar{u}_1, \dots, \bar{u}_m$, that is, if

$$\bar{v}=\lambda_1\bar{u}_1+\dots+\lambda_m\bar{u}_m,$$

where $\lambda_1, \dots, \lambda_m$ are any functions of $x_1, \dots, x_n, p_1, \dots, p_n$, then all $(m+1)$-rowed determinants of the matrix

$$\left\|\begin{matrix}\dfrac{\partial u_1}{\partial p_1}, & \cdot & \cdot & \dfrac{\partial u_1}{\partial p_n}, & \dfrac{\partial u_1}{\partial x_1}, & \cdot & \cdot & \dfrac{\partial u_1}{\partial x_n}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \dfrac{\partial u_m}{\partial p_1}, & \cdot & \cdot & \dfrac{\partial u_m}{\partial p_n}, & \dfrac{\partial u_m}{\partial x_1}, & \cdot & \cdot & \dfrac{\partial u_m}{\partial x_n}\\ \dfrac{\partial v}{\partial p_1}, & \cdot & \cdot & \dfrac{\partial v}{\partial p_n}, & \dfrac{\partial v}{\partial x_1}, & \cdot & \cdot & \dfrac{\partial v}{\partial x_n}\end{matrix}\right\|$$

must vanish identically; and therefore v must be a function of $u_1, \dots, u_m$.

Again, if u and v are any two functions of

$$x_1, \dots, x_n, p_1, \dots, p_n$$

we see that

$$\frac{\partial}{\partial x_i}(u, v) = \left(\frac{\partial u}{\partial x_i}, v\right) + \left(u, \frac{\partial v}{\partial x_i}\right),$$

$$\frac{\partial}{\partial p_i}(u, v) = \left(\frac{\partial u}{\partial p_i}, v\right) + \left(u, \frac{\partial v}{\partial p_i}\right);$$

and therefore, $\bar{u}$ and $\bar{v}$ being the corresponding operators, the alternant $\bar{u}\,\bar{v} - \bar{v}\,\bar{u}$ which is equal to

$$\sum^{i=n}\left(\bar{u}\frac{\partial v}{\partial p_i} - \bar{v}\frac{\partial u}{\partial p_i}\right)\frac{\partial}{\partial x_i} + \sum^{i=n}\left(\bar{v}\frac{\partial u}{\partial x_i} - \bar{u}\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial p_i}$$

$$= \sum^{i=n}\left(\left(u, \frac{\partial v}{\partial p_i}\right) - \left(v, \frac{\partial u}{\partial p_i}\right)\right)\frac{\partial}{\partial x_i} + \sum^{i=n}\left(\left(v, \frac{\partial u}{\partial x_i}\right) - \left(u, \frac{\partial v}{\partial x_i}\right)\right)\frac{\partial}{\partial p_i}$$

$$= \sum^{i=n}\left(\frac{\partial}{\partial p_i}(u, v)\right)\frac{\partial}{\partial x_i} - \sum^{i=n}\left(\frac{\partial}{\partial x_i}(u, v)\right)\frac{\partial}{\partial p_i}.$$

It follows that the alternant of $\bar{u}$ and $\bar{v}$ is derived from the function (u, v) by the rule which derived the operator $\bar{u}$ from the function u.

It is for this reason that we called the function (u, v) the alternant of the functions u and v; and what we have proved is expressed symbolically by

$$(\bar{u}, \bar{v}) = (\overline{u, v}).$$

If then u and v are in involution the operators $\bar{u}$ and $\bar{v}$ are commutative, and conversely.

§ 159. Let the operator $p_1\frac{\partial}{\partial p_1} + \ldots + p_n\frac{\partial}{\partial p_n}$ be denoted by P; we shall now prove that P is not connected with $u_1, \ldots, u_m$. Suppose it were so connected, then every $(m+1)$-rowed determinant of the matrix

$$\left\|\begin{matrix} \frac{\partial u_1}{\partial p_1}, & \cdot & \cdot & \frac{\partial u_1}{\partial p_n}, & \frac{\partial u_1}{\partial x_1}, & \cdot & \cdot & \frac{\partial u_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial u_m}{\partial p_1}, & \cdot & \cdot & \frac{\partial u_m}{\partial p_n}, & \frac{\partial u_m}{\partial x_1}, & \cdot & \cdot & \frac{\partial u_m}{\partial x_n} \\ 0, & \cdot & \cdot & 0, & p_1, & \cdot & \cdot & p_n \end{matrix}\right\|$$

would vanish identically.

It follows that every m-rowed determinant of the matrix

$$\left\| \begin{matrix} \dfrac{\partial u_1}{\partial p_1}, & \dots & \dfrac{\partial u_1}{\partial p_n} \\ \cdot & \cdot & \cdot \\ \dfrac{\partial u_m}{\partial p_1}, & \dots & \dfrac{\partial u_m}{\partial p_n} \end{matrix} \right\|$$

must vanish; there must therefore be some function of the form $\phi(u_1, \dots, u_m)$ which does not involve $p_1, \dots, p_n$. By passing to an equivalent function system we may take this function to be u_m, where u_m only involves $x_1, \dots, x_n$.

Every $(m+1)$-rowed determinant now vanishes in the matrix

$$\left\| \begin{matrix} \dfrac{\partial u_1}{\partial p_1}, & \dots & \dfrac{\partial u_1}{\partial p_n}, & \dfrac{\partial u_1}{\partial x_1}, & \dots & \dfrac{\partial u_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial u_{m-1}}{\partial p_1}, & \dots & \dfrac{\partial u_{m-1}}{\partial p_n}, & \dfrac{\partial u_{m-1}}{\partial x_1}, & \dots & \dfrac{\partial u_{m-1}}{\partial x_n} \\ 0, & \dots & 0, & \dfrac{\partial u_m}{\partial x_1}, & \dots & \dfrac{\partial u_m}{\partial x_n} \\ 0, & \dots & 0, & p_1, & \dots & p_n \end{matrix} \right\|.$$

Now u_m does not contain $p_1, \dots, p_n$, so that every two-rowed determinant of

$$\left\| \begin{matrix} \dfrac{\partial u_m}{\partial x_1}, & \dots & \dfrac{\partial u_m}{\partial x_n} \\ \cdot & \cdot & \cdot \\ p_1, & \dots & p_n \end{matrix} \right\|$$

cannot vanish; else would u_m be a mere constant, which is contrary to the hypothesis that $u_1, \dots, u_m$ are unconnected.

We must therefore conclude that every $(m-1)$-rowed determinant of

$$\left\| \begin{matrix} \dfrac{\partial u_1}{\partial p_1}, & \dots & \dfrac{\partial u_1}{\partial p_n} \\ \cdot & \cdot & \cdot \\ \dfrac{\partial u_{m-1}}{\partial p_1}, & \dots & \dfrac{\partial u_{m-1}}{\partial p_n} \end{matrix} \right\|$$

vanishes identically.

We now proceed as before, and passing to an equivalent

system to $u_1, \ldots, u_{m-1}$ may assume that u_{m-1} does not contain $p_1, \ldots, p_n$; and we thus see that either every $(m-2)$-rowed determinant of the matrix

$$\left\| \begin{matrix} \frac{\partial u_1}{\partial p_1}, & \cdot & \cdot & \cdot & \frac{\partial u_1}{\partial p_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial u_{m-2}}{\partial p_1}, & \cdot & \cdot & \cdot & \frac{\partial u_{m-2}}{\partial p_n} \end{matrix} \right\|$$

vanishes identically; or else every 3-rowed determinant of

$$\left\| \begin{matrix} \frac{\partial u_{m-1}}{\partial x_1}, & \cdot & \cdot & \cdot & \frac{\partial u_{m-1}}{\partial x_n} \\ \frac{\partial u_m}{\partial x_1}, & \cdot & \cdot & \cdot & \frac{\partial u_m}{\partial x_n} \\ p_1, & \cdot & \cdot & \cdot & p_n \end{matrix} \right\|$$

vanishes identically.

Since u_{m-1} and u_m are functions of $x_1, \ldots, x_n$ alone, we see, as before, that the latter hypothesis is untenable; proceeding with the alternate hypothesis, we ultimately come to the conclusion that our hypothesis of P being connected with $\overline{u}_1, \ldots, \overline{u}_m$ is untenable.

§ 160. If u is a homogeneous function of degree s in the variables $p_1, \ldots, p_n$ it can be at once verified that

$$(P, \overline{u}) = (s-1)\,\overline{u}.$$

The problem of finding a homogeneous function of degree zero, in involution with each of the m homogeneous functions $u_1, \ldots, u_m$ (themselves mutually in involution), and unconnected with these functions, is therefore equivalent to that of finding an integral of the complete system of $(m+1)$ unconnected equations

$$\overline{u}_1 f = 0, \ldots, \overline{u}_m f = 0, \quad Pf = 0,$$

which shall not be a mere function of $u_1, \ldots, u_m$.

There are $(2n-m-1)$ common integrals of

$$\overline{u}_1 f = 0, \ldots, \overline{u}_m f = 0, \quad Pf = 0;$$

if any one of the functions $u_1, \ldots, u_m$ is of zero degree then it will be an integral. There must, however, be at least

$(2n-m-1-m)$ common integrals unconnected with $u_1, \ldots, u_m$; and, as m is less than n, we can find at least one integral of zero degree unconnected with $u_1, \ldots, u_m$.

We now see how the complete integral of a given equation

$$f(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$$

is to be obtained.

We may write the equation in such a form as to give one of the variables in terms of the others; say in one of the forms

$$(1) \qquad x_1 = \phi_1(x_2, \ldots, x_n, p_1, \ldots, p_n),$$

$$\text{or,} \quad (2) \qquad p_1 = \phi_1(x_1, \ldots, x_n, p_2, \ldots, p_n).$$

We must then find, if we take the first form, a homogeneous function of zero degree in involution with $x_1 - \phi_1$, and unconnected with it; knowing then two homogeneous functions in involution, we find a third homogeneous function in involution with these two, unconnected with them, and of zero degree; proceeding thus, we finally obtain n unconnected functions in involution, one of which is $x_1 - \phi_1$.

If we equate each of these functions, except $x_1 - \phi_1$, to arbitrary constants, and $x_1 - \phi_1$ to zero, we shall have a Pfaffian system of equations which will include the given equation, and will involve $(n-1)$ arbitrary constants; the generating equations of this system will be a complete integral. If we had taken the second form we should have proceeded similarly.

§ 161. An equation of the form $f(x_1, \ldots, x_n) = 0$ would not ordinarily be called a differential equation; but considering Lie's extension of the definition of an integral it should be regarded as a particular form of the differential equation. If $f(x_1, \ldots, x_n) = 0$ is one of this class of differential equations, then any other unconnected equations of the form

$$f(x_1, \ldots, x_n) = 0, \ldots, f_{n-1}(x_1, \ldots, x_n) = 0$$

will with $f = 0$ form a Pfaffian system: any point on the locus $f = 0$ will be an integral of the equation $f = 0$. These integrals are also complete integrals; for the coordinates of any point on the locus $f = 0$ will involve $(n-1)$ arbitrary constants.

If the assigned differential equation is of the form

$$(1) \qquad p_1 f_1(x_1, \ldots, x_n) + \ldots + p_n f_n(x_1, \ldots, x_n) = 0,$$

we could also have 'point' integrals, the equations which define each point generating a M_{n-1}; these points, however, will in general be isolated points satisfying the equations $f_1 = 0, \ldots, f_n = 0$, and will not therefore be complete integrals.

Suppose that the equations $f_1 = 0, \ldots, f_n = 0$ are equivalent to a smaller number of equations, say

$$\phi_1(x_1, \ldots, x_n) = 0, \ldots, \phi_m(x_1, \ldots, x_n) = 0,$$

we should have an $(n-m)$-way locus in space, any point of which would be an integral of the given equation (1); these integrals, however, would not be complete, since they would only involve $(n-m)$ arbitrary constants.

§ 162. *Example.* Consider the equation

$$x_1 x_2 p_3^2 = x_3^2 p_1 p_2,$$

of which a complete integral is

$$a_1^2 x_1^2 + a_2^2 x_2^2 + a_1 a_2 x_3^2 + 1 = 0.$$

The corresponding Pfaffian system is

$$\frac{p_1}{a_1^2 x_1} = \frac{p_2}{a_2^2 x_2} = \frac{p_3}{a_1 a_2 x_3}, \quad a_1^2 x_1^2 + a_2^2 x_2^2 + a_1 a_2 x_3^2 + 1 = 0,$$

which may be thrown into the form

$$\frac{p_1}{x_1(p_1 x_1 + p_2 x_2 + p_3 x_3)} + a_1^2 = 0,$$

$$\frac{p_2}{x_2(p_1 x_1 + p_2 x_2 + p_3 x_3)} + a_2^2 = 0, \quad x_1 x_2 p_3^2 - x_3^2 p_1 p_2 = 0.$$

These equations define an ∞^2 of M_2's, each of which consists of points on a surface together with their corresponding tangent planes.

We shall now try whether the given equation can be satisfied by an ∞^2 of M_2's, each of which consists of points on a curve together with the infinity of tangent planes which can be drawn at each point of this curve.

Let the generating equations be

$$x_3 = f(x_1), \; x_2 = \phi(x_1),$$

then the third Pfaffian equation must be

$$p_1 + p_2 \phi'(x_1) + p_3 f'(x_1) = 0,$$

where f' denotes the differential coefficient of f with respect to its argument.

If $x_1 x_2 p_3^2 - x_3^2 p_1 p_2 = 0$ is to be connected with this Pfaffian system, we must have

$$x_1 \phi p_3^2 + f^2 \phi' p_2^2 + f^2 f' p_2 p_3 = 0$$

for all values of x_1, p_2, p_3; and therefore we must have $\phi = 0$ and $f' = 0$ for all values of the argument x_1.

From the third Pfaffian equation we now conclude that $p_1 = 0$; and therefore

$$x_2 = 0, \quad x_3 = \text{constant}, \quad p_1 = 0$$

will be an ∞ of M_2's satisfying the given differential equation; we do not, however, obtain an ∞^2 of the required class of M_2's.

Example. Find the complete integrals of

$$p_1 x_1 + \ldots + p_n x_n = 0$$

which are straight lines.

§ 163. As an example of an equation having no integral which is a curve, take

$$p_1^2 + p_2^2 + 2 p_1 p_3 x_1 + 2 p_2 p_3 x_2 + 2 p_3^2 x_1 x_2 = 0$$

(Forsyth, *Differential Equations*, § 202, Ex. 1).

If the Pfaffian system

$$x_3 = f(x_1), \quad x_2 = \phi(x_1), \quad p_1 + p_2 \phi'(x_1) + p_3 f'(x_1) = 0$$

were to satisfy this equation, we should have

$$\phi'^2 + 1 = 0, \quad f'^2 - 2 x_1 (f' - \phi) = 0, \quad \phi'(f' - x_1) + \phi = 0;$$

and, as these equations are inconsistent, we conclude that the given equation has no integral of the required form.

In order to obtain examples of equations having integrals in Lie's extended sense, it is only necessary to write down any equations

$$f_1(x_1, \ldots, x_n) = 0, \ldots, f_s(x_1, \ldots, x_n) = 0,$$

involving $(n-1)$ effective arbitrary constants, and then to complete the Pfaffian system.

Let

$$f_{s+1}(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0, \ldots, f_n(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$$

be the remaining equations of the system; if we eliminate

the arbitrary constants from the system we shall have a single equation between $x_1, \dots, x_n, p_1, \dots, p_n$; the complete integral of this equation will be

$$f_1(x_1, \dots, x_n) = 0, \dots, f_s(x_1, \dots, x_n) = 0.$$

Example. Take the equations

$$x_3 = ax_1 + b,\ x_4 = a^2 x_2 + c$$

where a, b, c are arbitrary constants; the other two Pfaffian equations will be

$$p_1 + ap_3 = 0,\ p_2 + a^2 p_4 = 0;$$

and therefore $p_1^2 p_4 + p_2 p_3^2 = 0$ is an equation with the complete integral

$$x_3 = ax_1 + b,\ x_4 = a^2 x_2 + c.$$

CHAPTER XV

COMPLETE SYSTEMS OF HOMOGENEOUS FUNCTIONS

§ 164. Let $u_1, \ldots, u_m$ be m unconnected homogeneous functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$. If we form the alternant of any two of these functions u_i and u_j we obtain the homogeneous function (u_i, u_j); if (u_i, u_j) is unconnected with $u_1, \ldots, u_m$ we add it to this system and have thus $(m+1)$ unconnected homogeneous functions. Proceeding thus, since there cannot be more than $2n$ unconnected homogeneous functions, we must ultimately obtain what we call a *complete system of homogeneous functions*; that is, a system of functions homogeneous in $p_1, \ldots, p_n$, and such that the alternant of any two functions of the system is connected with the functions of the system.

Let us now take $u_1, \ldots, u_m$ to be a complete homogeneous function system, so that we have

$$(u_i, u_j) = w_{ij}(u_1, \ldots, u_m) \qquad \binom{i = 1, \ldots, m}{j = 1, \ldots, m}.$$

The functions w_{ij} of the arguments $u_1, \ldots, u_m$ are called the *structure functions* of the complete system; and, since $(u_i, u_j) + (u_j, u_i) = 0$, we must have $w_{ij} + w_{ji} = 0$.

If $v_1, \ldots, v_m$ is a system of functions *equivalent* to $u_1, \ldots, u_m$ (that is, if for all values of the suffix i, v_i can be expressed in terms of $u_1, \ldots, u_m$, and u_i in terms of $v_1, \ldots, v_m$), then, though $v_1, \ldots, v_m$ may not each separately be homogeneous functions, we call $v_1, \ldots, v_m$ a homogeneous function system.

If then we are given a system of functions $v_1, \ldots, v_m$ of the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$, how are we to know whether or not the system is a homogeneous one?

Denoting by P the operator

$$p_1 \frac{\partial}{\partial p_1} + \ldots + p_n \frac{\partial}{\partial p_n},$$

we shall prove that *the necessary and sufficient conditions*

that the system may be homogeneous are that $Pv_1, \ldots, Pv_m$ *should each be connected with* $v_1, \ldots, v_m$, that is, each be expressible in terms of $v_1, \ldots, v_m$.

Firstly, the conditions are necessary; for if $u_1, \ldots, u_m$ are m functions homogeneous in $p_1, \ldots, p_n$ and respectively of degrees $s_1, \ldots, s_m$, and forming a system equivalent to $v_1, \ldots, v_m$, then Pv_i is connected with $u_1, \ldots, u_m$, $Pu_1, \ldots, Pu_m$. Now Pu_j is equal to $s_j u_j$, and therefore Pv_i is a function of $u_1, \ldots, u_m$, and so also a function of $v_1, \ldots, v_m$; we thus see that the conditions are necessary.

Secondly, these conditions are sufficient; for suppose that

$$Pv_i = f_i(v_1, \ldots, v_m), \qquad (i = 1, \ldots, m);$$

then if $f_1, \ldots, f_m$ are each identically equal to zero, $v_1, \ldots, v_m$ will be homogeneous functions of zero degree. If on the other hand these functions do not vanish identically, we can find $(m-1)$ unconnected functions of $v_1, \ldots, v_m$ such that they are each annihilated by

$$f_1 \frac{\partial}{\partial v_1} + \ldots + f_m \frac{\partial}{\partial v_m},$$

and therefore by P.

Let these functions be $u_1, \ldots, u_{m-1}$; they will be homogeneous functions of degree zero; we can then find one other function of $v_1, \ldots, v_m$ say u_m, unconnected with $u_1, \ldots, u_{m-1}$, and satisfying the equation

$$f_1 \frac{\partial u_m}{\partial v_1} + \ldots + f_m \frac{\partial u_m}{\partial v_m} = u_m,$$

and therefore satisfying the equation $Pu_m = u_m$.

The function u_m is therefore homogeneous of degree unity; and, as the system $u_1, \ldots, u_m$ is equivalent to $v_1, \ldots, v_m$, we conclude that the necessary conditions are also sufficient.

§ 165. If $u_1, \ldots, u_m$ are m unconnected functions of $x_1, \ldots, x_n, p_1, \ldots, p_n$, which may or may not be a homogeneous system, we say that the system is *complete* if the alternant of any two of the functions is connected with $u_1, \ldots, u_m$. If then we form the alternant of $f(u_1, \ldots, u_m)$ and $\phi(u_1, \ldots, u_m)$ (where f and ϕ are any two functional symbols) we see that this alternant is connected with $u_1, \ldots, u_m$, if $u_1, \ldots, u_m$ are the functions of a complete system. It at once follows that $v_1, \ldots, v_m$ being any system equivalent to $u_1, \ldots, u_m$, the one system is complete, if the other is complete.

We can now give a general definition of a complete homogeneous function system, as a system of m unconnected functions $u_1, \ldots, u_m$ such that

$$\begin{aligned}(u_i, u_j) &= w_{ij}(u_1, \ldots, u_m),\\ Pu_i &= w_i(u_1, \ldots, u_m),\end{aligned} \qquad \binom{i = 1, \ldots, m}{j = 1, \ldots, m}.$$

The functions $w_{ij}, \ldots, w_i, \ldots$ are the structure functions of the system; we can pass to any equivalent system $v_1, \ldots, v_m$, and in so doing we should change the form of the structure functions. Thus when we pass to an equivalent system in which $v_1, \ldots, v_{m-1}$ are homogeneous of degree zero, and v_m homogeneous, either of degree zero or of degree unity, we have $w_1, \ldots, w_{m-1}$ each zero, and w_m either zero or unity. The main problem to be considered in this chapter is how to pass to a system equivalent to $u_1, \ldots, u_m$ in which the structure functions may have the simplest possible form.

If each function $u_1, \ldots, u_m$ is homogeneous and of degree zero, then $f(u_1, \ldots, u_m)$ is homogeneous and of degree zero; and therefore every equivalent system has all its functions of degree zero. If such a system is complete, we shall now prove that it is in involution.

Since (u_i, u_j) is by hypothesis a function of $u_1, \ldots, u_m$, it is homogeneous and of degree zero; but u_i and u_j are each homogeneous of degree zero, and therefore their alternant is homogeneous and of degree minus unity. The only way of reconciling these two facts is by supposing that $(u_i, \ldots, u_j)$ is identically zero; that is, the system must be in involution.

§ 166. We shall, as in § 153, denote by $\overline{u}_i$ the operator

$$\frac{\partial u_i}{\partial p_1}\frac{\partial}{\partial x_1} + \ldots + \frac{\partial u_i}{\partial p_n}\frac{\partial}{\partial x_n} - \frac{\partial u_i}{\partial x_1}\frac{\partial}{\partial p_1} - \ldots - \frac{\partial u_i}{\partial x_n}\frac{\partial}{\partial p_n};$$

and by $(\overline{u}_i, \overline{u}_j)$ the alternant of $\overline{u}_i$ and $\overline{u}_j$. We have proved that this operator is derived from the alternant of the functions u_i and u_j by the rule which derived the operator $\overline{u}_i$ from the function u_i.

We have also proved (§ 159) that the operators $\overline{u}_1, \ldots, \overline{u}_m$ and P are unconnected. If we form the alternant of P and $\overline{u}_i$ we get

$$\begin{aligned}(P, \overline{u}_i) &= \sum^{j=n}\left(P\frac{\partial u_i}{\partial p_j}\right)\frac{\partial}{\partial x_j} - \sum^{j=n}\left(P\frac{\partial u_i}{\partial x_j}\right)\frac{\partial}{\partial p_i} + \sum^{j=n}\frac{\partial u_i}{\partial x_j}\frac{\partial}{\partial p_j},\\ &= \sum^{j=n}\frac{\partial}{\partial p_j}(Pu_i)\frac{\partial}{\partial x_j} - \sum^{j=n}\frac{\partial}{\partial x_j}(Pu_i)\frac{\partial}{\partial p_j} - \overline{u}_i;\end{aligned}$$

that is, the alternant of P and $\overline{u}_i$ is derived from the function $(P-1)\,u_i$ by the rule which derived $\overline{u}_i$ from u_i.

If then $u_1, \ldots, u_m$ are functions forming a complete system, the operators $\overline{u}_1, \ldots, \overline{u}_m$ form a complete system; and if $u_1, \ldots, u_m$ form a complete homogeneous system, $\overline{u}_1, \ldots, \overline{u}_m, P$ will be a complete system of $(m+1)$ unconnected operators.

The operators $\overline{u}_1, \ldots, \overline{u}_m$ form a complete sub-system of operators within the system $\overline{u}_1, \ldots, \overline{u}_m\, P$; and the alternants $(P, \overline{u}_1), \ldots, (P, \overline{u}_m)$ are each connected with $\overline{u}_1, \ldots, \overline{u}_m$. From these facts we conclude that the complete system of equations

$$(u_1, f) = 0, \ldots, (u_m, f) = 0,$$

admits the infinitesimal transformation

$$p_i' = p_i + tp_i, \qquad (i = 1, \ldots, n);$$

and therefore, *if f is any function annihilated by $\overline{u}_1, \ldots, \overline{u}_m$, Pf will also be annihilated by these operators.*

§ 167. We shall now prove an important identity which will immediately be required.

If u, v, w are any three functions of the variables

$$x_1, \ldots, x_n,\ p_1, \ldots, p_n,$$

then it will be proved that

$$\big(u, (v, w)\big) + \big(w, (u, v)\big) + \big(v, (w, u)\big) \equiv 0.$$

Since $$\overline{(u, v)} = (\overline{u}, \overline{v})$$

it follows that the operator derived from

$$(1) \qquad \big(u, (v, w)\big) + \big(w, (u, v)\big) + \big(v, (w, u)\big)$$

is $$\big(\overline{u}, (\overline{v}, \overline{w})\big) + \big(\overline{w}, (\overline{u}, \overline{v})\big) + \big(\overline{v}, (\overline{w}, \overline{u})\big).$$

Now by Jacobi's identity this operator vanishes identically and therefore (1) must be a mere constant. We next prove that this constant is zero.

If we notice that

$$(uv, w) \equiv u\,(v, w) + v\,(u, w),$$

we may easily verify that

$$\begin{aligned}&\big(u^2, (v, w)\big) + \big(w, (u^2, v)\big) + \big(v, (w, u^2)\big)\\ &\equiv 2u\,\big[\big(u, (v, w)\big) + \big(w, (u, v)\big) + \big(v, (w, u)\big)\big];\end{aligned}$$

and we therefore conclude that the constant $= 2u \times$ some constant.

Now u being any function whatever of the variable, this

can only be true if the constants are zero; and therefore we see that

$$\big(u, (v, w)\big) + \big(w, (u, v)\big) + \big(v, (w, u)\big) \equiv 0.$$

Another proof of this theorem is given in Forsyth, *Differential Equations*, § 214.

Let now $u_1, \ldots, u_m$ be a complete system; then $\overline{u}_1, \ldots, \overline{u}_m$, being unconnected, there must be $(2n-m)$ unconnected functions of the variables which will be annihilated by $\overline{u}_1, \ldots, \overline{u}_m$. Let these functions be $v_1, \ldots, v_{2n-m}$; we must now have $(u_i, v_j) = 0$ for all values of the suffixes.

From the identity

$$\big(u_i, (v_j, v_k)\big) + \big(v_k, (u_i, v_j)\big) + \big(v_j, (v_k, u_i)\big) \equiv 0$$

we conclude that $\big(u_i (v_j, v_k)\big) \equiv 0$, and therefore $\overline{u}_i (v_j, v_k) \equiv 0$. We therefore have the theorem: every alternant of $v_1, \ldots, v_{2n-m}$ is annihilated by the operators $\overline{u}_1, \ldots, \overline{u}_m$.

Now every function annihilated by these operators will be connected with $v_1, \ldots, v_{2n-m}$; and therefore every alternant of $v_1, \ldots, v_{2n-m}$ is connected with this given set of functions; that is, $v_1, \ldots, v_{2n-m}$ is itself a complete function system.

The m unconnected functions $u_1, \ldots, u_m$ are annihilated by each of the $(2n-m)$ operators $\overline{v}_1, \ldots, \overline{v}_{2n-m}$, so that the two systems are reciprocally related, and each is said to be the *polar* of the other.

If $\overline{u}_1, \ldots, \overline{u}_m$ is a complete homogeneous system its polar system is also homogeneous. For $\overline{u}_1, \ldots, \overline{u}_m$ is homogeneous, and v_i is annihilated by $\overline{u}_1, \ldots, \overline{u}_m$; therefore (by § 166) Pv_i is also annihilated by $\overline{u}_1, \ldots, \overline{u}_m$; Pv_i must therefore be a function of $v_1, \ldots, v_{2n-m}$; that is, $v_1, \ldots, v_{2n-m}$ is a homogeneous function system.

Suppose that we are given a system $u_1, \ldots, u_m$ such that

$$\begin{aligned} (u_i, u_j) &= w_{ij} (u_1, \ldots, u_m), \\ Pu_i &= w_i \ (u_1, \ldots, u_m), \end{aligned} \qquad \binom{i = 1, \ldots, m}{j = 1, \ldots, m};$$

any function whatever of $u_1, \ldots, u_m$ will be a function of the system, but we regard $u_1, \ldots, u_m$ as the fundamental set of functions of the system once we have chosen them; if we were to change to an equivalent set of fundamental functions we should have to change the structure functions.

§ 168. It must now be proved that the functions which are common to a system and its polar system—that is, the functions which are connected with $u_1, \ldots, u_m$ and also with

$v_1, \ldots, v_{2n-m}$—will themselves form a homogeneous system in involution.

Let $u_1, \ldots, u_{m+q}$ be a complete homogeneous system; by properly choosing the fundamental functions of the system we may suppose that $u_{m+1}, \ldots, u_{m+q}$ are the functions of the given system which belong also to the polar system.

Since $u_{m+1}, \ldots, u_{m+q}$ are each annihilated by $\bar{u}_1, \ldots, \bar{u}_{m+q}$ they are functions in involution; and, since both the given system and its polar system are homogeneous,

$$Pu_{m+1}, \ldots, Pu_{m+q}$$

must be functions common to the two systems, and therefore must be functions of $u_{m+1}, \ldots, u_{m+q}$; that is, $u_{m+1}, \ldots, u_{m+q}$ is itself a homogeneous system.

We call this homogeneous sub-system of $u_1, \ldots, u_{m+q}$ its *Abelian sub-system*: if the Abelian sub-system coincides with the polar system, we say that the given system is a *satisfied* one.

If a system is satisfied its polar system is then a system in involution; conversely, if a system is in involution, its polar system is satisfied; for, if $v_1, \ldots, v_{2n-m}$ is a system in involution, all of these functions must also be contained in the polar system $u_1, \ldots, u_m$, which is therefore satisfied.

§ 169. Let $u_1, \ldots, u_m$ be a complete homogeneous system which is not satisfied; its polar system is, we know, a homogeneous one; but all the functions $v_1, \ldots, v_{2n-m}$ cannot be of zero degree, else would the polar system be in involution, and $u_1, \ldots, u_m$ a satisfied system. The polar system can then be thrown into such a form that v_1 is of degree unity, and $v_2, \ldots, v_{2n-m}$ each of zero degree; and it can therefore be thrown into such a form that each of its fundamental set of functions is of degree unity; for $v_1, v_1 v_2, \ldots, v_1 v_{2n-m}$ would be $(2n-m)$ unconnected functions of the polar system, each of degree unity.

Since $u_1, \ldots, u_m$ is not satisfied, not all of the functions $v_1, \ldots, v_{2n-m}$ of the polar system can be connected with $u_1, \ldots, u_m$. We may therefore suppose that v_1 is not so connected; and, as it is a homogeneous function of degree unity in involution with $u_1, \ldots, u_m$, we see that

$$u_1, \ldots, u_m, v_1$$

is a complete homogeneous function system of order $(m+1)$. Every unsatisfied system is therefore contained, as a sub-

system, within another complete homogeneous system whose order is greater by unity than that of the given system.

We thus see that we can continue to add new functions to a given system, till it will finally be contained as a sub-system, within a satisfied system.

§ 170. If we have two complete systems $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ with the same structure functions; that is, if

$$(u_i, u_j) = w_{ij}(u_1, \ldots, u_m), \; Pu_i = w_i(u_1, \ldots, u_m),$$
$$(v_i, v_j) = w_{ij}(v_1, \ldots, v_m), \; Pv_i = w_i(v_1, \ldots, v_m),$$

then, if one system is satisfied, so is the other.

To prove this consider the linear operator

$$w_{i1}(u_1, \ldots, u_m)\frac{\partial}{\partial u_1} + \ldots + w_{im}(u_1, \ldots, u_m)\frac{\partial}{\partial u_m},$$

which we call the *contracted* operator of $\overline{u}_i$. Let $f(u_1, \ldots, u_m)$ be any function of $u_1, \ldots, u_m$; then, since

$$\overline{u}_i f(u_1, \ldots, u_m) = (u_i, u_1)\frac{\partial f}{\partial u_1} + \ldots + (u_i, u_m)\frac{\partial f}{\partial u_m},$$

we see that the contracted operator of $\overline{u}_i$ has the same effect on any function of $u_1, \ldots, u_m$ as the operator $\overline{u}_i$ itself.

The contracted operator of P is

$$w_1(u_1, \ldots, u_m)\frac{\partial}{\partial u_1} + \ldots + w_m(u_1, \ldots, u_m)\frac{\partial}{\partial u_m}.$$

The Abelian sub-system of $u_1, \ldots, u_m$ consists of the functions annihilated by the contracted operators of $\overline{u}_1, \ldots, \overline{u}_m$.

If $u_1, \ldots, u_m$ is a satisfied system, every function annihilated by $\overline{u}_1, \ldots, \overline{u}_m$ is also annihilated by the contracted operators; and therefore there are $(2n-m)$ functions of $u_1, \ldots, u_m$ which are annihilated by the contracted operators. Since the contracted operators of $\overline{v}_1, \ldots, \overline{v}_m$ are of exactly the same form in $v_1, \ldots, v_m$ that the contracted operators of $\overline{u}_1, \ldots, \overline{u}_m$ are in $u_1, \ldots, u_m$, it follows that there are $(2n-m)$ unconnected functions of $v_1, \ldots, v_m$ annihilated by the contracted operators of $\overline{v}_1, \ldots, \overline{v}_m$; and therefore $v_1, \ldots, v_m$ is also a satisfied system.

If $u_1, \ldots, u_m$ is an unsatisfied system, we have proved that a homogeneous function u_{m+1} can be added to it, such that u_{m+1} is of degree unity, and in involution with $u_1, \ldots, u_m$. If then we have two systems $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$, with the same structure functions, we can add u_{m+1} to the first, and v_{m+1} to the second, in such a way that $u_1, \ldots, u_{m+1}$ and

$v_1, \dots, v_{m+1}$ will still remain homogeneous function systems of like structure.

We thus see that if we are given two systems $u_1, \dots, u_m$ and $v_1, \dots, v_m$, of like structure, we can add functions to each, in such a way that the new systems become satisfied simultaneously, and have, when both satisfied, still the same structure.

§ 171. We must now show how a complete homogeneous system is to be reduced to its simplest form.

We first find the Abelian sub-system of the given system $u_1, \dots, u_m, u_{m+1}, \dots, u_{m+q}$; to find this it is only necessary to form the contracted operators of $\overline{u}_1, \dots, \overline{u}_{m+q}$, and then to find the functions of $u_1, \dots, u_{m+q}$ which these annihilate. We may now suppose that the fundamental functions have been so chosen that $u_{m+1}, \dots, u_{m+q}$ is this Abelian sub-system; and we further suppose that each of the functions $u_1, \dots, u_{m+q}$ are given in homogeneous form, so that u_i is of degree s_i, in the variables $p_1, \dots, p_n$.

Since the contracted operator of $\overline{u}_i$ is

$$(u_i, u_1)\frac{\partial}{\partial u_1} + \dots + (u_i, u_{m+q})\frac{\partial}{\partial u_{m+q}},$$

we see that the contracted operators of $\overline{u}_{m+1}, \dots, \overline{u}_{m+q}$ vanish identically.

The contracted operator of $\overline{u}_j$, where j does not exceed m, is

$$(u_j, u_1)\frac{\partial}{\partial u_1} + \dots + (u_j, u_m)\frac{\partial}{\partial u_m}, \qquad (j = 1, \dots, m),$$

and these contracted operators of $\overline{u}_1, \dots, \overline{u}_m$ cannot be connected. For if they were connected, they would form a complete system of operators in $u_1, \dots, u_m$, and would therefore have at least one common integral which would be a function of $u_1, \dots, u_m$. Now this integral, being a function annihilated by $\overline{u}_1, \dots, \overline{u}_{m+q}$, would be an Abelian function of the group, which would be contrary to our hypothesis that $u_{m+1}, \dots, u_{m+q}$ are the only unconnected Abelian functions in the system.

The contracted operator of P is

$$s_1 u_1 \frac{\partial}{\partial u_1} + \dots + s_{m+q} u_{m+q} \frac{\partial}{\partial u_{m+q}},$$

and we have (as proved for the more general case in § 159),

$$(1) \qquad P\overline{u}_i - \overline{u}_i P = (s_i - 1)\,\overline{u}_i.$$

We have proved that we may take the functions of the system in such a form that they are either all homogeneous of degree zero, or all but one of degree zero, and that one of degree unity.

In the first case the functions are all in involution and the system cannot be thrown into any simpler form.

In the second case the function of degree unity may be an Abelian function, or it may be a non-Abelian function of the system.

We consider these alternatives; and we first suppose that the Abelian function u_{m+1} is of degree unity, and that $u_1, \ldots, u_m, u_{m+2}, \ldots, u_{m+q}$ are each of degree zero.

§ 172. Each of the alternants $(u_1, u_2), \ldots, (u_1, u_m)$ will now be of degree minus unity, and therefore

$$u_{m+1}(u_1, u_2), \ldots, u_{m+1}(u_1, u_m)$$

will each be homogeneous functions of degree zero; and, as they are functions of $u_1, \ldots, u_{m+q}$, all of which except u_{m+1} are of zero degree, we conclude that they are functions of $u_1, \ldots, u_m, u_{m+2}, \ldots, u_{m+q}$ only.

It now follows that some function of

$$u_1, \ldots, u_m, u_{m+2}, \ldots, u_{m+q}$$

can be found, say $f(u_1, \ldots, u_m, u_{m+2}, \ldots, u_{m+q})$, such that

$$u_{m+1} \cdot \overline{u}_1 f = 1;$$

and therefore (since $\overline{u}_1 u_{m+1} = 0$) $u_{m+1} f$ will be a function of $u_1, \ldots, u_{m+q}$, of degree unity in $p_1, \ldots, p_m$, and such that

$$\overline{u}_1 u_{m+1} f = 1.$$

Since $u_{m+1} f$ cannot be an Abelian function of the system (else would it be in involution with u_1, and annihilated by $\overline{u}_1$), we may therefore take the functions of the fundamental system in such a form that u_2 and also u_{m+1} are of unit degree, whilst all the other functions are of degree zero; $(u_1, u_2) = 1$, and $u_{m+1}, \ldots, u_{m+q}$ are Abelian functions.

Since $(u_1, u_2) = 1$, $\overline{u}_1$ and $\overline{u}_2$ will be permutable, and therefore the contracted operators of $\overline{u}_1$ and $\overline{u}_2$ will also be permutable. There are therefore $(m+q-2)$ unconnected functions of $u_1, \ldots, u_{m+q}$ annihilated by $\overline{u}_1$ and $\overline{u}_2$; and, from the formula (1) of § 171, we see that if $f(u_1, \ldots, u_{m+q})$ is one such function $Pf(u_1, \ldots, u_{m+q})$ will be another such. These functions therefore form a complete homogeneous function

system in themselves; and, since $(u_1, u_2) = 1$, each one of these functions must be unconnected with u_1 and u_2.

It follows from the above discussion that we may take the fundamental functions of the system in such a form that u_1 and u_2 are in involution with $u_3, \ldots, u_{m+q}$; that

$$u_{m+1}, \ldots, u_{m+q}$$

are Abelian functions, and $(u_1, u_2) = 1$; and further that u_2 and u_{m+1} are each of degree unity, whilst the other functions are of degree zero.

Since $u_3, \ldots, u_{m+q}$ is now in itself a complete homogeneous function system, we may treat it in a similar manner, and thus reduce the function system to the form

$$u_1, v_1, u_2, v_2, \ldots, u_s, v_s, v_{s+1}, \ldots, v_{s+q},$$

where $u_1, \ldots, u_s, v_{s+2}, \ldots, v_{s+q}$ are each homogeneous of zero degree, and $v_1, \ldots, v_{s+1}$ are each homogeneous of degree unity; and where further

$$(u_1, v_1) = (u_2, v_2) = \ldots = (u_s, v_s) = 1,$$

all other alternants of the system vanishing identically.

If instead of the functions $v_{s+1}, \ldots, v_{s+q}$, we take the Abelian functions $v_{s+1}, v_{s+1} v_{s+2}, \ldots, v_{s+1} v_{s+q}$, we obtain the *normal* form. In this all the functions $u_1, \ldots, u_s$ are of degree zero, all the functions $v_1, \ldots, v_{s+q}$ are of degree unity, and

$$\text{(A)} \qquad (u_1, v_1) = (u_2, v_2) = \ldots = (u_s, v_s) = 1,$$

while all the other alternants of the system vanish identically.

§ 173. We next take the case where all the Abelian functions are of degree zero, and we take u_1 to be of degree unity, whilst all other functions of the system are of zero degree.

Since $$(u_1, u_2), \ldots, (u_1, u_m)$$

are each homogeneous functions of degree zero, they must be functions of $u_2, \ldots, u_{m+q}$ only; and we can therefore find a homogeneous function of degree zero, say $f(u_2, \ldots, u_{m+q})$, such that

$$\overline{u}_1 \cdot f = 1.$$

We now see as in the last article that we may take the functions of the system to be

$$u_1, u_2, u_3, \ldots, u_{m+1}, \ldots, u_{m+q},$$

where $(u_1, u_2) = 1$, and all the other functions are in involution with these two, and form in themselves a complete homogeneous function system.

The system $u_3, \ldots, u_{m+q}$ cannot have all its functions of degree zero, else would these functions all be Abelian within the system $u_1, \ldots, u_{m+q}$, which is contrary to the hypothesis that there were only q such functions.

We may therefore, since the Abelian functions are each of degree zero, take u_3 to be of degree unity.

We then, as before, reduce this system to the normal form

$$\text{(B)} \qquad u_1, v_1, u_2, v_2, \ldots, u_s, v_s, v_{s+1}, \ldots, v_{s+q},$$

where $u_1, \ldots, u_s$ are homogeneous of degree unity, and $v_1, \ldots, v_{s+q}$ homogeneous of degree zero, and where

$$(u_1, v_1) = (u_2, v_2) = \ldots = (u_s, v_s) = 1,$$

whilst all the other alternants vanish identically.

Every complete homogeneous system is therefore such that all its functions are of degree zero, and therefore all its alternants vanish identically; or it is equivalent to one of the two forms (A), or (B).

§ 174. It is important to notice that, in bringing $u_1, \ldots, u_m$ to normal form, we replace these functions by an equivalent system of fundamental functions

$$f_1(u_1, \ldots, u_m), \ldots, f_m(u_1, \ldots, u_m);$$

and to find the forms of the functions $f_1, \ldots, f_m$ we did not make use of the operators $\overline{u}_1, \ldots, \overline{u}_m$ themselves, but only of the contracted forms of these operators, viz.

$$(u_i, u_1)\frac{\partial}{\partial u_1} + \ldots + (u_i, u_m)\frac{\partial}{\partial u_m} \qquad (i = 1, \ldots, m).$$

If therefore $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ are two complete homogeneous systems of like structure, and, if

$$f_1(u_1, \ldots, u_m), \ldots, f_m(u_1, \ldots, u_m)$$

is a system equivalent to $u_1, \ldots, u_m$ and in normal form, then

$$f_1(v_1, \ldots, v_m), \ldots, f_m(v_1, \ldots, v_m)$$

will be a function system equivalent to $v_1, \ldots, v_m$, and will be in normal form.

§ 175. We can now prove that a complete homogeneous system, which contains Abelian functions, is contained as

a sub-system within a larger system, not containing any Abelian functions.

We take the system in normal form (A)

$$u_1, \ldots, u_s, v_1, \ldots, v_{s+q},$$

where $v_1, \ldots, v_{s+q}$ are each of degree unity.

The functions $u_1, \ldots, u_s, v_s, v_{s+2}, \ldots, v_{s+q}$ now form a system complete in itself; if we form the system polar to this it must contain v_{s+1}; but in the polar system v_{s+1} cannot be an Abelian function, since it is not a function of the system $u_1, \ldots, u_s, v_s, v_{s+2}, v_{s+q}$.

We can therefore find within the dual system a homogeneous function of degree zero, say u_{s+1}, such that

$$(u_{s+1}, v_{s+1}) = 1.$$

We now have the homogeneous system

$$u_1, \ldots, u_{s+1}, v_1, \ldots, v_{s+q},$$

which is of normal form but only contains $(q-1)$ Abelian functions. Proceeding similarly, we finally obtain a system of $(2s+2q)$ homogeneous functions

$$u_1, \ldots, u_{s+q}, v_1, \ldots, v_{s+q},$$

such that

$$(u_1, v_1) = (u_2, v_2) = \ldots = (u_{s+q}, v_{s+q}) = 1,$$

and all other alternants vanish identically; $u_1, \ldots, u_{s+q}$ are each homogeneous of degree unity; $v_1, \ldots, v_{s+q}$ are each homogeneous of zero degree; and there are in the system no Abelian functions; that is, no functions in involution with all functions of the system.

We should obtain the same results had we taken systems of either of the normal forms

$$u_1, \ldots, u_s, v_1, \ldots, v_{s+q},$$

where $v_1, \ldots, v_{s+q}$ are each functions of degree zero; or

$$v_1, \ldots, v_m,$$

where $v_1, \ldots, v_m$ are all of degree zero, and therefore all in involution.

§ 176. In a satisfied system, since the polar system is now the Abelian sub-system, $q = 2n-2s-q$, and therefore

$$2s+2q = 2n;$$

if then we apply this reasoning to a satisfied system we see

that it is contained in a system of order $2n$, which has no Abelian functions.

As we have proved that every complete system is contained as a sub-system within a satisfied system, we see that every system is a sub-system within a homogeneous system of order $2n$.

If $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ are two complete homogeneous systems of the same structure, we can then take, as a fundamental set of functions of the first, a system

$$f_1(u_1, \ldots, u_m), \ldots, f_m(u_1, \ldots, u_m);$$

and as the fundamental functions of the second

$$f_1(v_1, \ldots, v_m), \ldots, f_m(v_1, \ldots, v_m),$$

and we can add functions to each of these systems, till finally we have two function systems, of order $2n$, which will be in normal form, will contain no Abelian functions, and will be of the same structure, with $f_i(u_1, \ldots, u_m)$ corresponding to $f_i(v_1, \ldots, v_m)$.

CHAPTER XVI

CONTACT TRANSFORMATIONS

§ 177. We know (§ 154) that if $X_1, \ldots, X_n$ are functions of

$$x_1, \ldots, x_n, p_1, \ldots, p_n,$$

homogeneous and of zero degree in $p_1, \ldots, p_n$, the necessary and sufficient conditions, in order that

$$X_1 = a_1, \ldots, X_n = a_n$$

may be a Pfaffian system of equations, for all values of the constants $a_1, \ldots, a_n$, are that $X_1, \ldots, X_n$ should be unconnected functions in involution. It follows that $p_1 dx_1 + \ldots + p_n dx_n$ will be a sum of multiples of $dX_1, \ldots, dX_n$ if, and only if, $X_1, \ldots, X_n$ are unconnected functions, in involution, and homogeneous in $p_1, \ldots, p_n$ of zero degree.

If then we know n unconnected functions $X_1, \ldots, X_n$ satisfying these conditions, n other functions $P_1, \ldots, P_n$ of the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ can be found such that

$$P_1 dX_1 + \ldots + P_n dX_n = p_1 dx_1 + \ldots + p_n dx_n.$$

Let us now seek the conditions in order that

$$x'_i = X_i, \ p'_i = P_i, \qquad (i = 1, \ldots, n),$$

where $X_1, \ldots, X_n$, $P_1, \ldots, P_n$ are unknown functions of

$$x_1, \ldots, x_n, p_1, \ldots, p_n,$$

may lead to the equation

$$\sum^{i=n} p'_i dx'_i = \sum^{i=n} p_i dx_i.$$

Consider the Pfaffian equation

$$\sum^{i=n} p_i dx_i - \sum^{i=n} p'_i dx'_i = 0$$

in the $4n$ unconnected variables

$$x_1, \ldots, x_n, p_1, \ldots, p_n, x'_1, \ldots, x'_n, p'_1, \ldots, p'_n.$$

The necessary and sufficient conditions that the $2n$ equations

$$(1) \quad x'_i - X_i = 0, \quad p'_i - P_i = 0, \qquad (i = 1, \ldots, n)$$

should satisfy it are the three following.

Firstly, the equations must be unconnected; this condition is evidently satisfied since $x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ are unconnected.

Secondly, the equations

$$x'_i - X_i = 0, \qquad (i = 1, \ldots, n)$$

must be homogeneous of zero degree in $p_1, \ldots, p_n, p'_1, \ldots, p'_n$; and therefore $X_1, \ldots, X_n$ must each be homogeneous of zero degree in $p_1, \ldots, p_n$. Similarly we see that $P_1, \ldots, P_n$ must each be homogeneous of the first degree in $p_1, \ldots, p_n$.

Thirdly, the equations must be in involution. It is easily seen that the following identities hold for all forms of the unknown functions $X_1, \ldots, X_n, P_1, \ldots, P_n$, viz.

$$(x'_i - X_i,\ x'_k - X_k) = (X_i, X_k),$$
$$(x'_i - X_i,\ p'_k - P_k) = (x'_i, p'_k) + (X_i, P_k) = (X_i, P_k) \text{ if } i \neq k,$$
$$(x'_i - X_i,\ p'_i - P_i) = (x'_i, p'_i) + (X_i, P_i) = -1 + (X_i, P_i),$$
$$(p'_i - P_i,\ p'_k - P_k) = (P_i, P_k).$$

If then the given equations are in involution, we must have, for all values of $x_1, \ldots, x_n, p_1, \ldots, p_n, x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ satisfying the equations (1),

$$(X_i, X_k) = 0,\ (X_i, P_k) = 0 \text{ if } i \neq k,\ (X_i, P_i) = 1,\ (P_i, P_k) = 0.$$

Now from the given equations (1) no equation connecting $x_1, \ldots, x_n, p_1, \ldots, p_n$ can be deduced; and therefore the given equations cannot be in involution, unless we have identically

$$(X_i, X_k) = 0,\ (X_i, P_k) = 0 \text{ if } i \neq k,\ (X_i, P_i) = 1,\ (P_i, P_k) = 0.$$

We therefore have the following important theorem:

$$x'_i = X_i,\ p'_i = P_i, \qquad (i = 1, \ldots, n)$$

will then, and then only, lead to

$$\sum_{i=1}^{i=n} p'_i\, dx'_i = \sum_{i=1}^{i=n} p_i\, dx_i;$$

that is, to the identity

$$\sum_{i=1}^{i=n} P_i\, dX_i = \sum_{i=1}^{i=n} p_i\, dx_i,$$

if X_i is homogeneous, and of zero degree in $p_1, \ldots, p_n$, P_i is homogeneous, and of the first degree in $p_1, \ldots, p_n$, and

$$(X_i, X_k) = 0,\ (X_i, P_k) = 0 \text{ if } i \neq k,\ (X_i, P_i) = 1,\ (P_i, P_k) = 0.$$

It must now be proved that there cannot be any functional connexion between $X_1, \ldots, X_n$, $P_1, \ldots, P_n$.

§ 178. Suppose that it were possible to express P_n in the form
$$P_n = V(X_1, \ldots, X_n, P_1, \ldots, P_{n-1}),$$
where V is some functional symbol; then we should have
$$(X_n, V) = (X_n, P_n) = 1;$$
and, since X_n is in involution with $X_1, \ldots, X_n, P_1, \ldots, P_{n-1}$, it must be in involution with V, and therefore (X_n, V) would be equal both to zero and to unity.

There cannot then be any connexion between $X_1, \ldots X_n$, $P_1, \ldots, P_n$ involving any of the functions $P_1, \ldots, P_n$. Suppose that there could be a functional connexion between $X_1, \ldots, X_n$ alone; then, since the equations
$$X_1 = a_1, \ldots, X_n = a_n$$
(where $a_1, \ldots, a_n$ are any constants) satisfy Pfaff's equation
$$p_1 dx_1 + \ldots + p_n dx_n = 0,$$
we know from § 154 that the given equations must be unconnected; and this result is inconsistent with the hypothesis of $X_1, \ldots, X_n$ being connected.

We conclude then that $X_1, \ldots, X_n, P_1, \ldots, P_n$ are entirely unconnected; and therefore
$$(1) \qquad x'_i = X_i,\ p'_i = P_i, \qquad (i = 1, \ldots, n)$$
will be a transformation scheme since by means of this equation system we can express each of the variables $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ in terms of $x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$.

The transformation scheme (1) is said to be a *homogeneous contact transformation scheme,* since it does not alter the Pfaffian expression, but transforms
$$\sum^{i=n} p_i dx_i \text{ into } \sum^{i=n} p'_i dx'_i.$$

The scheme we are considering transforms elements in space $x_1, \ldots, x_n$ into elements in space $x'_1, \ldots, x'_n$; and, if two consecutive elements of the one space are united, the corresponding elements of the other space will be united. The danger of a geometrical misinterpretation must be guarded against: thus, if A is a point in one space and a a plane through A, the point and the plane together make up an element of that space; if B is a second point in the same space and β a plane

through it then we have a second element in the same space. Let now A' be the point in the other space which corresponds to the element A, a (not merely to the point A) and a' the plane through A' corresponding to the same element; and let B' and β' have similar meanings with respect to B, β. If B lies on a it is not at all necessary that B' should lie on a'. If, however, B is contiguous to A, and β to a, then B, β is a contiguous element to A, a; and, if B lies on a, they are united elements; we then see (the transformation scheme between the elements being a contact one), that B' lies on a', and A' on β', and that B', β' and A', a' are united elements.

§ 179. It is important to notice that the contact transformation scheme is altogether known when we know the functions $X_1, \dots, X_n$. To prove this let the known functions, homogeneous, of zero degree in $p_1, \dots, p_n$, and in involution, be $X_1, \dots, X_n$. We have proved that functions $P_1, \dots, P_n$ must exist such that

$$P_1\,dX_1 + \dots + P_n\,dX_n = p_1\,dx_1 + \dots + p_n\,dx_n,$$

and therefore by the reasoning of § 178,

$$X_1, \dots, X_n,\ P_1, \dots, P_n$$

will be unconnected, and

$$x_i' = X_i,\ p_i' = P_i, \qquad (i = 1, \dots, n)$$

will be a homogeneous contact transformation.

That the functions $P_1, \dots, P_n$ are known, when $X_1, \dots, X_n$ are known, follows from the equations

$$\sum^{i=n} P_i \frac{\partial X_i}{\partial x_k} = p_k, \quad \sum^{i=n} P_i \frac{\partial X_i}{\partial p_k} = 0, \qquad (k = 1, \dots, n).$$

These equations could only then fail to determine $P_1, \dots, P_n$ uniquely in terms of $x_1, \dots, x_n, p_1, \dots, p_n$ when all n-rowed determinants of the matrix

$$\left\| \begin{matrix} \dfrac{\partial X_1}{\partial x_1}, & \cdot\ \cdot & \dfrac{\partial X_1}{\partial x_n}, & \dfrac{\partial X_1}{\partial p_1}, & \cdot\ \cdot & \dfrac{\partial X_1}{\partial p_n} \\ \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \dfrac{\partial X_n}{\partial X_1}, & \cdot\ \cdot & \dfrac{\partial X_n}{\partial x_n}, & \dfrac{\partial X_n}{\partial p_1}, & \cdot\ \cdot & \dfrac{\partial X_n}{\partial p_n} \end{matrix} \right\|$$

vanish identically, that is, when $X_1, \dots, X_n$ are connected; and as $X_1, \dots, X_n$ are unconnected the equations do not fail.

The problem then of finding a homogeneous contact transformation is that of finding n unconnected functions of zero degree in $p_1, \ldots, p_n$, and mutually in involution; and to every such system of functions one contact transformation scheme will correspond. We have shown in Chapter XIV how this problem depends on the solution of a complete system of linear partial differential equations of the first order; and we have also seen how, when we are given m of the n functions in involution, the remaining $(n-m)$ are to be found.

Example. Any n unconnected functions of $x_1, \ldots, x_n$ are in involution and of zero degree; the contact transformation scheme, however, which corresponds to this solution of the problem, will be a mere point transformation.

If on the other hand we take any $(n-1)$ unconnected functions of $p_1, \ldots, p_n$ of zero degree they will be in involution; as there cannot be more than $(n-1)$ such functions the n^{th} function of the involution system must involve $x_1, \ldots, x_n$.

Let us take $\dfrac{p_1}{p_n}, \ldots, \dfrac{p_{n-1}}{p_n}$ as the $(n-1)$ functions; and let v be the n^{th} function; since it is in involution with $\dfrac{p_1}{p_n}$ we must have $\dfrac{1}{p_n}\dfrac{\partial v}{\partial x_1} - \dfrac{p_1}{p_n^2}\dfrac{\partial v}{\partial x_n} = 0$; we therefore have the following equations to determine v:

$$\frac{\dfrac{\partial v}{\partial x_1}}{p_1} = \frac{\dfrac{\partial v}{\partial x_2}}{p_2} = \ldots = \frac{\dfrac{\partial v}{\partial x_n}}{p_n},$$

and may take v to be the function

$$\frac{p_1 x_1 + \ldots + p_n x_n}{p_n}.$$

We now have n unconnected functions in involution, and of zero degree, viz.

$$X_1 = \frac{p_1}{p_n}, \ldots, X_{n-1} = \frac{p_{n-1}}{p_n}, \quad X_n = \frac{p_1 x_1 + \ldots + p_n x_n}{p_n}.$$

The identity

$$P_1 dX_1 + \ldots + P_n dX_n = p_1 dx_1 + \ldots + p_n dx_n$$

gives us

$$\sum^{i=n-1} (P_i + x_i P_n)\, d\left(\frac{p_i}{p_n}\right) + P_n \sum^{i=n} \frac{p_i}{p_n} dx_i = \sum^{i=n} p_i\, dx_i;$$

and therefore

$$P_n = p_n, \quad P_1 = -x_1 p_n, \ldots, P_{n-1} = -x_{n-1} p_n.$$

We thus have the homogeneous contact transformation

$$x'_1 = \frac{p_1}{p_n}, \ldots, x'_{n-1} = \frac{p_{n-1}}{p_n}, \quad x'_n = \frac{p_1 x_1 + \ldots + p_n x_n}{p_n},$$

$$p'_1 = -x_1 p_n, \ldots, p'_{n-1} = -x_{n-1} p_n, \quad p'_n = p_n.$$

§ 180. By a homogeneous contact transformation any Pfaffian system is transformed into a Pfaffian system. For if

(1) $f_1(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0, \ldots, f_n(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$

are the equations of a Pfaffian system; the contact transformation

(2) $x'_i = X_i, \; p'_i = P_i, \qquad (i = 1, \ldots, n)$

will transform these equations into some other n equations, say

(3) $\phi_1(x'_1, \ldots, x'_n, p'_1, \ldots, p'_n) = 0, \ldots, \phi_n(x'_1, \ldots, x'_n, p'_1, \ldots, p'_n) = 0.$

What we have therefore to prove is, that any consecutive values of $x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ satisfying the equations (3) will satisfy the equation

$$p'_1 \, dx'_1 + \ldots + p'_n \, dx'_n = 0.$$

Now to two consecutive values of $x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ satisfying (3), there will correspond two consecutive values of $x_1, \ldots, x_n, p_1, \ldots, p_n$ satisfying (1); and therefore—from the definition of a Pfaffian system—satisfying the equation

$$p_1 \, dx_1 + \ldots + p_n \, dx_n = 0.$$

Since the transformation is a homogeneous contact one

$$p'_1 \, dx'_1 + \ldots + p'_n \, dx'_n = p_1 \, dx_1 + \ldots + p_n \, dx_n = 0;$$

and therefore the equations (3) satisfy the definition of a Pfaffian system.

If we know any integral of an assigned differential equation of the first order, we know how to write down a Pfaffian system which will include the assigned differential equation. If to this known Pfaffian system we apply any known homogeneous contact transformation, the assigned differential equation will be transformed into another equation, of which we shall know the Pfaffian system, and therefore the integral.

It is at this point that we begin to see the advantage of Lie's extended definition of an integral of a given equation.

The assigned differential equation may only have an ordinary integral, that is, the Pfaffian system, which contains it, may have only one generating equation; yet possibly the equation into which the differential equation is transformed will have, as the Pfaffian system including it, one generated by two or more equations.

It may even happen that by the contact transformation the assigned differential equation is transformed into an equation only containing $x_1', \ldots, x_n'$, that is, into a generating equation of the Pfaffian system.

§ 181. *Example.* Consider the equation

$$2x_2 x_3^2 p_1^2 = x_1^2 p_2 p_4$$

of which a complete integral is easily found, viz.

$$bx_3^2 + cx_3 + x_2^2 x_3 - ax_1^2 + 4a^2 x_3 x_4 = 0,$$

where a, b, c are arbitrary constants.

If $f(x_1, \ldots, x_n) = 0$ is an integral of an assigned differential equation $\phi(x_1, \ldots, x_n, p_1, \ldots, p_n) = 0$, then this integral gives us the Pfaffian system

$$\frac{p_1}{\frac{\partial f}{\partial x_1}} = \frac{p_2}{\frac{\partial f}{\partial x_2}} = \ldots = \frac{p_n}{\frac{df}{dx_n}}, \qquad f = 0;$$

and, since from the definition of an integral, $\phi = 0$ is deducible from these n equations, it must be one of the equations of the system.

In the example before us it is then only necessary to add two equations to the given differential equation and its integral, in order to have a Pfaffian system; the third equation which we could obtain would be connected with these four.

We may take these equations to be

$$2ax_3 p_1 + x_1 p_4 = 0$$

and
$$4a^2 x_3 p_3 - (x_2^2 + c + 2bx_3 + 4a^2 x_4) p_4 = 0,$$

and, by aid of the given integral, the second of these is thrown into the more convenient form

$$4a^2 x_3^2 p_3 - (bx_3^2 + ax_1^2) p_4 = 0.$$

The Pfaffian system with which we are now concerned is then

$$(1) \qquad 2x_2 x_3^2 p_1^2 - x_1^2 p_2 p_4 = 0,$$

$$(2) \qquad bx_3^2 + cx_3 + x_2^2 x_3 - ax_1^2 + 4a^2 x_3 x_4 = 0,$$

(3) $$2ax_3p_1+x_1p_4=0,$$

(4) $$4a^2x_3^2p_3-(bx_3^2+ax_1^2)p_4=0.$$

If we apply to this system the contact transformation,

$$x_1'=\frac{p_1}{p_4},\quad x_2'=\frac{p_2}{p_4},\quad x_3'=\frac{p_3}{p_4},\quad x_4'=\frac{p_1}{p_4}x_1+\frac{p_2}{p_4}x_2+\frac{p_3}{p_4}x_3+x_4,$$

$$p_1'=-x_1p_4,\quad p_2'=-x_2p_4,\quad p_3'=-x_3p_4,\quad p_4'=p_4,$$

we obtain the four Pfaffian equations

(1) $$p_1'^2p_4'x_2'+2p_2'p_3'^2x_1'^2=0,$$

(2) $$bp_3'^2-cp_3'p_4'-p_2'^2\frac{p_3'}{p_4'}-ap_1'^2$$
$$-4a^2p_3'(p_1'x_1'+p_2'x_2'+p_3'x_3'+p_4'x_4')=0,$$

(3) $2ap_3'x_1'+p_1'=0,$ (4) $4a^2p_3'^2x_3'-bp_3'^2-ap_1'^2=0.$

Eliminating p_1', p_2', p_3', p_4' from these equations, we obtain, after a little labour, not one but two equations, viz.

$$4a^3x_1'^2-4a^2x_3'+b=0,\quad c-4a^4x_2'+4a^2x_4'=0.$$

It follows, therefore, that by the contact transformation we pass from the equation

$$2x_2x_3^2p_1^2-x_1^2p_2p_4=0,$$

with its ordinary complete integral

$$bx_3^2+cx_3+x_2^2x_3-ax_1^2+4a^2x_3x_4=0,$$

to the equation $\quad p_1^2p_4x_2+2p_2p_3^2x_1^2=0,$

with Lie's complete integral

$$4a^3x_1^2-4a^2x_3+b=0,\quad c-4a^4x_2+4a^2x_4=0.$$

Example. Any equation of the form

$$p_1x_1+\ldots+p_nx_n=p_nf\left(\frac{p_1}{p_n},\ldots,\frac{p_{n-1}}{p_n}\right)$$

is transformed by the contact transformation

$$x_1'=\frac{p_1}{p_n},\ldots,x_{n-1}'=\frac{p_{n-1}}{p_n},\quad x_n'=\frac{p_1x_1+\ldots+p_nx_n}{p_n},$$

$$p_1'=-x_1p_n,\ldots,p_{n-1}=-x_{n-1}p_n,\quad p_n'=p_n$$

into $$x_n'=f(x_1',\ldots,x_{n-1}').$$

This would not be a differential equation at all, according

to the usual definition, but is one in Lie's sense; and, since we know a complete integral of it, viz.

$$x'_1 = a_1, \ldots, x'_n = a_n,$$

where $a_1, \ldots, a_n$ are constants connected by the law

$$a_n = f(a_1, \ldots, a_{n-1}),$$

we at once deduce that

$$a_1 x_1 + \ldots + a_{n-1} x_{n-1} + x_n = f(a_1, \ldots, a_{n-1})$$

is a complete integral of the given equation.

§ 182. The functions $X_1, \ldots, X_n, P_1, \ldots, P_n$ which define a homogeneous contact transformation satisfy the conditions of being a complete homogeneous system of functions in normal form; for

$$(X_1, P_1) = (X_2, P_2) = \ldots = (X_n, P_n) = 1,$$

and all other alternants of the system vanish identically; whilst $X_1, \ldots, X_n$ are homogeneous of degree zero, and $P_1, \ldots, P_n$ homogeneous of degree unity.

If we are given two homogeneous function systems of like structure

$$u_1, \ldots, u_m \text{ and } v_1, \ldots, v_m,$$

we must now prove that they can be transformed, the one into the other, by a homogeneous contact transformation.

If $$f_1(u_1, \ldots, u_m), \ldots, f_m(u_1, \ldots, u_m)$$

are functions equivalent to $u_1, \ldots, u_m$, and such that $f_1, \ldots, f_m$ are in normal form, we know that

$$f_1(v_1, \ldots, v_m), \ldots, f_m(v_1, \ldots, v_m)$$

will be a function system equivalent to $v_1, \ldots, v_m$, and of the same normal form as

$$f_1(u_1, \ldots, u_m), \ldots, f_m(u_1, \ldots, u_m).$$

Also if a contact transformation

$$x'_i = X_i, \; p'_i = P_i, \qquad (i = 1, \ldots, n)$$

transforms $f_j(v_1, \ldots, v_m)$ into $f_j(u_1, \ldots, u_m)$ for all values of the suffix j from 1 to m, that is, if

$$f_j(v'_1, \ldots, v'_m) = f_j(u_1, \ldots, u_m), \qquad (j = 1, \ldots, m),$$

where v'_j denotes the same function of $x'_1, \ldots, x'_n, p'_1, \ldots, p'_n$ that v_j is of $x_1, \ldots, x_n, p_1, \ldots, p_n$, then will

$$v'_j = u_j, \qquad (j = 1, \ldots, m).$$

In order, therefore, to prove that two homogeneous function systems of like structure are transformable into one another by a homogeneous contact transformation, it will only be necessary to prove that two such systems of the same normal form are so transformable.

We have seen that to $u_1, \ldots, u_m$ we can add functions $u_{m+1}, \ldots, u_{2n}$, till $u_1, \ldots, u_{2n}$ is a system of order $2n$, containing no Abelian functions, and in normal form; these $2n$ functions will therefore define a homogeneous contact transformation scheme. If we similarly add functions to the system $v_1, \ldots, v_m$ till it forms a complete homogeneous system of order $2n$, containing no Abelian functions, and in normal form, then $v_1, \ldots, v_{2n}$ will also define a homogeneous contact transformation scheme.

In these two systems u_i is homogeneous and of the same degree in $p_1, \ldots, p_n$ that v_i is, viz. unity or zero; and when we say that the two systems have like structure we mean that u_i in one system corresponds to v_i in the other.

We may suppose that $u_1, \ldots, u_n$ are the functions of zero degree, and $u_{n+1}, \ldots, u_{2n}$ the functions of degree unity;

$$x_i'' = u_i, \; p_i'' = u_{n+i}, \qquad (i = 1, \ldots, n)$$

will then lead to

$$\sum^{i=n} p_i'' dx_i'' = \sum^{i=n} p_i \, dx_i;$$

and $$x_i'' = v_i', \; p_i'' = v_{n+i}', \qquad (i = 1, \ldots, n)$$

will lead to $$\sum^{i=n} p_i'' dx_i'' = \sum^{i=n} p_i' \, dx_i'.$$

It follows that the equations

$$u_i = v_i', \qquad (i = 1, \ldots, 2n)$$

will lead to $$\sum^{i=n} p_i' \, dx_i' = \sum^{i=n} p_i \, dx_i;$$

that is, the functions $v_1, \ldots, v_{2n}$ are transformable to the functions $u_1, \ldots, u_{2m}$ by a homogeneous contact transformation scheme; and in particular $v_1, \ldots, v_m$ are transformable into $u_1, \ldots, u_m$, v_i being transformed into u_i.

§ 183. Having now proved that two complete homogeneous systems of the same order and structure are transformable into one another by a homogeneous contact transformation,

we shall now investigate the conditions under which it is possible to transform *any* m given functions $v_1, \ldots, v_m$ respectively into the given functional forms $u_1, \ldots, u_m$, by a homogeneous contact transformation.

Let $\qquad x'_i = X_i,\ p'_i = P_i, \qquad (i = 1, \ldots, n)$

be a homogeneous contact transformation; we have

$$\left.\begin{aligned}\frac{\partial}{\partial p_k} &= \sum^{i=n} \frac{\partial X_i}{\partial p_k}\frac{\partial}{\partial x'_i} + \sum^{i=n} \frac{\partial P_i}{\partial p_k}\frac{\partial}{\partial p'_i},\\ \frac{\partial}{\partial x_k} &= \sum^{i=n} \frac{\partial X_i}{\partial x_k}\frac{\partial}{\partial x'_i} + \sum^{i=n} \frac{\partial P_i}{\partial x_k}\frac{\partial}{\partial p'_i},\end{aligned}\right\} \quad (k = 1, \ldots, n).$$

Suppose that this contact transformation transforms v_j into u_j, where

$v_j = \phi_j(x_1, \ldots, x_n, p_1, \ldots, p_n)$ and $u_j = f_j(x_1, \ldots, x_n, p_1, \ldots, p_n)$,

so that

$$f_j(x_1, \ldots, x_n, p_1, \ldots, p_n) = \phi_j(x'_1, \ldots, x'_n, p'_1, \ldots, p'_n);$$

then

$$\bar{u}_j = \sum^{i=k=n} (X_i, X_k)\frac{\partial v'_j}{\partial x'_i}\frac{\partial}{\partial x'_k} + \sum^{i=k=n} (X_i, P_k)\frac{\partial v'_j}{\partial x'_i}\frac{\partial}{\partial p'_k}$$

$$+ \sum^{i=k=n} (I_i, X_k)\frac{\partial v'_j}{\partial p'_i}\frac{\partial}{\partial x'_k} + \sum^{i=k=n} (P_i, P_k)\frac{\partial v'_j}{\partial p'}\frac{\partial}{\partial p'_k};$$

that is, by the conditions for a homogeneous contact transformation,

$$\bar{u}_j = \sum^{i=n} \frac{\partial v'_j}{\partial p'_i}\frac{\partial}{\partial x'_i} - \sum^{i=n} \frac{\partial v'_j}{\partial x'_i}\frac{\partial}{\partial p'_i} = \bar{v}'_j.$$

From the mere fact that $u_j = v'_j$ we could not of course conclude that $\bar{u}_j = \bar{v}'_j$; we were only able to draw this conclusion from the forms of the functions $X_1, \ldots, X_n, P_1, \ldots, P_n$ which define the homogeneous contact transformation.

Since $\bar{u}_j = \bar{v}'_j$, and $u_i = v'_i$,

$$\bar{u}_j \cdot u_i = \bar{v}'_j \cdot v'_i; \text{ and therefore } (u_j, u_i) = (v'_j, v'_i);$$

and therefore the transformation, which transforms $v_1, \ldots, v_m$ into $u_1, \ldots, u_n$ respectively, must transform the alternant (v_i, v_j) into the alternant (u_i, u_j).

Again since

$$\frac{\partial}{\partial p_k} = \sum^{i=n} \frac{\partial X_i}{\partial p_k}\frac{\partial}{\partial x_i'} + \sum^{i=n} \frac{\partial P_i}{\partial p_k}\frac{\partial}{\partial p_i'}, \qquad (k = 1, \ldots, n),$$

$$P = \sum^{k=n} p_k \frac{\partial}{\partial p_k} = \sum^{i=k=n} p_k \frac{\partial X_i}{\partial p_k}\frac{\partial}{\partial x_i'} + \sum^{i=k=n} p_k \frac{\partial P_i}{\partial p_k}\frac{\partial}{\partial p_i'};$$

and, as $X_1, \ldots, X_n$ are of zero degree, and $P_1, \ldots, P_n$ of degree unity, we therefore have

$$P = \sum^{i=n} P_i \frac{\partial}{\partial p_i'} = \sum^{i=n} p_i' \frac{\partial}{\partial p_i'} = P'.$$

The transformation then which transforms v_i into u_i must also transform Pv_i into Pu_i.

From these considerations we see that, given the functions $v_1, \ldots, v_m$ and $u_1, \ldots, u_m$, we must form the complete homogeneous systems of which they are respectively functions. To do this form the alternants from $v_1, \ldots, v_m$ and also the functions $Pv_1, \ldots, Pv_m$; if by this means we obtain no function unconnected with $v_1, \ldots, v_m$ the system is complete and homogeneous; if, on the other hand, we obtain a new function we add it to $v_1, \ldots, v_m$, and proceed similarly with the new system. As there cannot be more than $2n$ unconnected functions of $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ we must thus ultimately arrive at a complete homogeneous function system. When we have formed the two complete homogeneous systems of lowest orders which contain the given sets of functions, we can tell whether or not the systems are of the same order and structure; if they are, the given functions $v_1, \ldots, v_m$ are respectively transformable into $u_1, \ldots, u_m$ by a homogeneous contact transformation, but otherwise they are not so transformable.

Thus any homogeneous function can be transformed into any other of the same degree; for the function group of each is of order one, and the structure the same.

In particular, any homogeneous function u of degree unity can be transformed into p_1; and therefore the operator $\overline{u}$ can be transformed into $\frac{\partial}{\partial x_1}$ by a homogeneous contact transformation if, and only if, u is of degree unity.

So if $u_1, \ldots, u_m$ are m unconnected homogeneous functions, each of degree unity and mutually in involution, they can be

transformed into $p_1, \dots, p_m$, and therefore $\overline{u}_1, \dots, \overline{u}_m$ can be transformed into $\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_m}$ respectively.

§ 184. Although in considering the theory of Pfaffian systems of equations it is much more convenient to work with the homogeneous equation

$$p_1 dx_1 + \dots + p_n dx_n = 0,$$

yet in particular examples, and in the cases $n = 2$, and $n = 3$, it is often simpler to take the non-homogeneous form

$$dz = p_1 dx_1 + \dots + p_n dx_n. \tag{1}$$

It is clear that to satisfy this equation we must have at least $(n+1)$ unconnected equations between

$$z, x_1, \dots, x_n, p_1, \dots, p_n,$$

but instead of considering this equation independently we may deduce its theory from that of the corresponding homogeneous equation.

Let $$z = y_{n+1}, \quad x_1 = y_1, \dots, x_n = y_n,$$

$$p_1 = -\frac{q_1}{q_{n+1}}, \dots, p_n = -\frac{q_n}{q_{n+1}}, \tag{2}$$

where q_{n+1} is not zero; then the equation (1) is equivalent to the homogeneous one

$$q_1 dy_1 + \dots + q_{n+1} dy_{n+1} = 0.$$

To satisfy this equation we must have $(n+1)$ unconnected equations in $y_1, \dots, y_{n+1}, q_1, \dots, q_{n+1}$; and in order that

$$Y_1 = a_1, \dots, Y_{n+1} = a_{n+1}$$

may satisfy the equation, for all values of the arbitrary constants, it is necessary and sufficient that $Y_1, \dots, Y_{n+1}$ should be $(n+1)$ unconnected homogeneous functions of

$$y_1, \dots, y_{n+1}, q_1, \dots, q_{n+1}$$

in involution.

Let Z be the function in $z, x_1, \dots, x_n, p_1, \dots, p_n$ equivalent to Y_{n+1}; and $X_1, \dots, X_n$ the functions which correspond to $Y_1, \dots, Y_n$ respectively.

If F is any function of $y_1, \dots, y_{n+1}, \dfrac{q_1}{q_{n+1}}, \dots, \dfrac{q_n}{q_{n+1}}$, it is also a function of $x_1, \dots, x_n, z, p_1, \dots, p_n$, in which form we shall

denote it by Φ: we then have $F = \Phi$, and

$$\frac{\partial F}{\partial y_{n+1}} = \frac{\partial \Phi}{\partial z}, \quad \frac{\partial F}{\partial q_{n+1}} = -\frac{1}{q_{n+1}} \sum_{i=1}^{i=n} p_i \frac{\partial \Phi}{\partial p_i},$$

$$\frac{\partial F}{\partial Y_i} = \frac{\partial \Phi}{\partial x_i}, \quad \frac{\partial F}{\partial q_i} = -\frac{1}{q_{n+1}} \frac{\partial \Phi}{\partial p_i}, \qquad (i = 1, \ldots, n).$$

If we now denote the expression

$$\sum^{i=n} \frac{\partial u}{\partial p_i}\left(\frac{\partial v}{\partial x_i} + p_i \frac{\partial v}{\partial z}\right) - \sum^{i=n} \frac{\partial v}{\partial p_i}\left(\frac{\partial u}{\partial x_i} + p_i \frac{\partial u}{\partial z}\right)$$

by $[u, v]_{z,x,p}$, we deduce that

$$\left.\begin{aligned} (Y_i, Y_k)_{y,q} &= -\frac{1}{q_{n+1}}[X_i, X_k], \\ (Y_{n+1}, Y_k)_{y,q} &= -\frac{1}{q_{n+1}}[Z, X_k], \end{aligned}\right. \qquad \binom{i = 1, \ldots, n}{k = 1, \ldots, n}.$$

We conclude therefore that the necessary and sufficient conditions, in order that

$$Z = a_{n+1}, \; X_1 = a_1, \ldots, X_n = a_n$$

may satisfy the equation

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

are $\quad [Z, X_i] = 0, \; [X_i, X_k] = 0, \qquad \binom{i = 1, \ldots, n}{k = 1, \ldots, n}.$

If two functions u and v of the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ are such that $[u, v]_{z,x,p} = 0$, we say they are in involution. Similarly we say that two equations $u = 0$, $v = 0$ are in involution if the equation $[u, v]_{z,x,p} = 0$ is connected with $u = 0$, $v = 0$. We generally omit the suffixes, and write $[u, v]$ for $[u, v]_{z,x,p}$, the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ being understood.

The equations $\quad Z = 0, \; X_1 = 0, \ldots, X_n = 0,$

will then, and only then, satisfy the Pfaffian equation

$$dz = p_1 dx_1 + \ldots + p_n dx_n,$$

when they are unconnected and in involution.

It follows that $(n+1)$ unconnected equations in involution cannot all be equations in $x_1, \ldots, x_n, p_1, \ldots, p_n$ only, but must contain z; else would they not lead to

$$dz = p_1 dx_1 + \ldots + p_n dx_n.$$

We may prove this last result independently thus; suppose

$$Z = 0,\ X_1 = 0, \dots, X_n = 0$$

do not contain z, we then see that

$$[Z, X_i] = (Z, X_i) \text{ and } [X_i, X_k] = (X_i, X_k);$$

we now have the $(n+1)$ unconnected differential equations

$$(Z, f) = 0,\ (X_1, f) = 0, \dots, (X_n, f) = 0,$$

with the $(n+1)$ unconnected integrals

$$X_1 - a_1 = 0, \dots, X_n - a_n = 0, Z - a_{n+1} = 0,$$

and this is impossible, the equations being in $2n$ variables only.

§ 185. Suppose we have $(n+1)$ unconnected functions of $z, x_1, \dots, x_n, p_1, \dots, p_n$ in involution, viz. $Z, X_1, \dots, X_n$. If we apply the transformation (2) of § 184, the identities

$$[Z, X_i] = 0,\ [X_i, X_k] = 0$$

are transformed to

$$(Y_i, Y_j) = 0, \qquad \begin{pmatrix} i = 1, \dots, n+1 \\ j = 1, \dots, n+1 \end{pmatrix}.$$

We have therefore $(n+1)$ unconnected functions of

$$y_1, \dots, y_{n+1}, q_1, \dots, q_{n+1},$$

homogeneous and of zero degree in $q_1, \dots, q_{n+1}$, and in involution. We can therefore write down the homogeneous contact transformation

$$y'_i = Y_i,\ q'_i = Q_i, \qquad (i = 1, \dots, n+1);$$

and, since
$$\sum^{i=n+1} q'_i\, dy'_i = \sum^{i=n+1} q_i\, dy_i,$$
we see that, if

$$P_i = -\frac{Q_i}{Q_{n+1}}, \qquad (i = 1, \dots, n),$$

$$dy'_{n+1} - P_1\, dy'_1 - \dots - P_n\, dy'_n = \frac{q_{n+1}}{Q_{n+1}} (dz - p_1\, dx_1 - \dots - p_n\, dx_n).$$

Therefore

$$(1) \quad z' = Z,\ x'_i = X_i,\ p'_i = P_i \qquad (i = 1, \dots, n)$$

will be a transformation, with the property

$$dz' - p'_1\, dx'_1 - \dots - p'_n\, dx'_n = \rho\,(dz - p_1\, dx_1 - \dots - p_n\, dx_n),$$

where $\rho = \frac{q_{n+1}}{Q_{n+1}}$, and is therefore a homogeneous function of $y_1, \ldots, y_{n+1}, q_1, \ldots, q_{n+1}$, of zero degree, and therefore a function of $z, x_1, \ldots, x_n, p_1, \ldots, p_n$.

A transformation such as (1) is called a *contact transformation*; and we see that, when we are given the $(n+1)$ unconnected functions in involution, viz. $Z, X_1, \ldots, X_n$, the contact transformation is entirely given.

The functions $P_1, \ldots, P_n$, as well as the factor ρ, may be obtained algebraically from the equations

$$\frac{\partial Z}{\partial z} - \sum_{}^{i=n} P_i \frac{\partial X_i}{\partial z} = \rho,$$

$$\frac{\partial Z}{\partial x_k} - \sum_{}^{i=n} P_i \frac{\partial X_i}{\partial x_k} = -\rho p_k, \qquad (k = 1, \ldots, n),$$

$$\frac{\partial Z}{\partial p_k} - \sum_{}^{i=n} P_i \frac{\partial X_i}{\partial p_k} = 0. \qquad (k = 1, \ldots, n).$$

The contact transformation

$$z' = Z, \quad x'_i = X_i, \quad p'_i = P_i, \qquad (i = 1, \ldots, n)$$

has the property of leaving the Pfaffian equation

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0$$

unaltered; and therefore—from the general definition of a group—the system of all contact transformations, regarded as transformation schemes in the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, generates a continuous group, though of course not a finite continuous group.

§ 186. *Example.* The variables being $y_1, y_2, y_3, q_1, q_2, q_3$, and u_1, u_2, v_1, v_2 being unconnected homogeneous functions of zero degree, such that every function of u_1, u_2 is in involution with every function of v_1, v_2, but u_1 not in involution with u_2, nor v_1 with v_2, it is required to find simple forms to which these functions may be reduced by a contact transformation.

The alternant (u_1, u_2) is of degree minus unity, and cannot therefore be a function of u_1 and u_2; we have therefore three unconnected functions u_1, u_2, and (u_1, u_2); and, as v_1 is in involution with u_1 and u_2, it is also in involution with (u_1, u_2). We thus see that u_1, u_2 and (u_1, u_2) are three unconnected functions of a homogeneous system; and that there are at least three unconnected functions in involution

with each of these functions, viz. v_1, v_2 and (v_1, v_2); it therefore follows (since the number of variables is six) that there cannot be more than three functions in the system containing u_1, u_2 and (u_1, u_2). The conclusion we draw is that u_1, u_2, (u_1, u_2) form a complete homogeneous function system, and that v_1, v_2, (v_1, v_2) is its polar system.

Since u_1, u_2, (u_1, u_2) is a system of order three, it must have at least one Abelian function. We see this by recalling the normal form of a complete system; or we may prove it independently by writing down the contracted operators of a complete system of order three, when, since the Pfaffian determinant

$$\begin{vmatrix} 0 \quad , & (u_1, u_2), & (u_1, u_3) \\ (u_2, u_1), & 0 \quad , & (u_2, u_3) \\ (u_3, u_1), & (u_3, u_2), & 0 \end{vmatrix}$$

vanishes identically, we see that not more than two of the contracted operators can be unconnected.

If all two-rowed minors of the above determinant vanished, then all the functions would be in involution; there must therefore be either three or only one Abelian function.

In this example, since (u_1, u_2) is not zero, there must be one, and only one, Abelian function; and, as it is not a mere function of u_1 and u_2 (for then u_1 and u_2 would be in involution), it is not of zero degree (see § 165). When the system is then reduced to normal form it is of like structure with

$$y_1, \; q_1, \; q_3;$$

and can therefore be reduced to this form by a homogeneous contact transformation.

We can therefore, by a homogeneous contact transformation, so reduce u_1 and u_2 that each will be a homogeneous function of y_1, q_1, q_3 of zero degree.

Since v_1 and v_2 are homogeneous functions of zero degree, in involution with every function of u_1 and u_2, they are in involution with y_1 and $\dfrac{q_1}{q_3}$. Since they are in involution with y_1, they cannot involve q_1; and, since they are also in involution with $\dfrac{q_1}{q_3}$, we see that they cannot involve y_1 or y_3. We conclude therefore that v_1 and v_2 are homogeneous functions of y_2, q_2, q_3 of zero degree.

If we now take

$$z = y_3, \; x_1 = y_1, \; x_2 = y_2, \; p_1 = \frac{-q_1}{q_3}, \; p_2 = \frac{-q_2}{q_3},$$

we see that u_1 and u_2 can be transformed by a contact transformation so as to be functions of x_1 and p_1; while by the same contact transformations v_1 and v_2 become functions of x_2 and p_2

§ 187. The above example has an important application to Ampère's equation,

$$Rr + Ss + Tt + U(rt - s^2) = V.$$

If this equation admits the two systems of intermediary integrals

$$u_1 = f(u_2) \text{ and } v_1 = \phi(v_2)$$

(where f and ϕ are arbitrary functional symbols), then we know (Forsyth, *Differential Equations*, § 237) that

$$[u_1, v_2] = 0, \ [u_1, v_1] = 0, \ [u_2, v_1] = 0, \ [u_2, v_2] = 0.$$

From what we have proved, we see that, when we have applied a suitable contact transformation to the original variables, we may take u_1 and u_2 to be functions of x and p only. Now by a contact transformation any equation of Ampère's form is transformed into some other equation of the same form. In the new variables then, Ampère's equation has an intermediary integral

$$u_1 = f(u_2),$$

where u_1 and u_2 do not involve y, z, or q.

This equation is therefore to be the result of eliminating the arbitrary function from

$$\frac{\partial u_1}{\partial x} + r\frac{\partial u_1}{\partial p} = f'(u_2)\left(\frac{\partial u_2}{\partial x} + r\frac{\partial u_2}{\partial p}\right),$$

$$s\frac{\partial u_1}{\partial p} = sf'(u_2)\frac{\partial u_2}{\partial p}.$$

The eliminant is

$$s\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial p} - \frac{\partial u_1}{\partial p}\frac{\partial u_2}{\partial x}\right) = 0;$$

and, as u_1 is not a function of u_2, we cannot have

$$\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial p} - \frac{\partial u_1}{\partial p}\frac{\partial u_2}{\partial x} = 0,$$

so that the equation must be $s = 0$. This is therefore the

form to which an equation of Ampère, admitting the two systems of intermediary integrals

$$u_1 - f(u_2) = 0 \text{ and } v_1 - \phi(v_2) = 0,$$

can be reduced by a contact transformation.

An interesting proof of this theorem of Lie's is given in Goursat, *Équations aux dérivées partielles du second ordre*, I. p. 39.

If in the equation $Rr + Ss + Tt + U(rt - s^2) = V$ we have $S^2 = 4(RT + UV)$, there can only be one system of intermediary integrals, $u_1 = f(u_2)$. We now have, however, $[u_1, u_2] = 0$; for, since the roots are equal in the equation

$$\lambda^2(RT + UV) - \lambda US + U^2 = 0,$$

we have (Forsyth, *Differential Equations*, § 238) $u_1 = v_2$; and, since $[u_1, v_2] = 0$, we must therefore have in the limiting case $[u_1, u_2] = 0$.

We now take $u_1 = p$, $u_2 = q$; and we see that $p = f(q)$ can only be an intermediary integral for all forms of the function if the equation is

$$(rt - s^2) = 0.$$

This then is the form to which this class of Ampèrian equation, with the intermediary integral $u_1 - f(u_2) = 0$, can be reduced by a contact transformation.

CHAPTER XVII

THE GEOMETRY OF CONTACT TRANSFORMATIONS

§ 188. If the equations defining a contact transformation are

$$(1) \qquad z' = Z,\ x'_i = X_i,\ p'_i = P_i, \qquad (i = 1, \ldots, n),$$

we know that the $(n+1)$ functions $X_1, \ldots, X_n, Z$ form a system in involution; and conversely, when we are given any involution system, we know how to construct a contact transformation scheme.

In this chapter we shall show how contact transformation schemes may be constructed without previously constructing involution systems.

If we eliminate $p_1, \ldots, p_n$ from the $(n+1)$ equations (1), we shall obtain at least one equation of the form

$$f(x_1, \ldots, x_n, z, x'_1, \ldots, x'_n, z') = 0;$$

and we may obtain 1, 2, ..., $(n+1)$ such equations. We call these equations the *generating* equations of the contact transformation scheme.

Suppose that we have s generating equations, viz.

$$f_1 = 0, \ldots, f_s = 0,$$

then the equation

$$(2) \qquad dz' - \sum^{i=n} p'_i\, dx'_i - \rho\,(dz - \sum^{i=n} p_i\, dx_i) = 0$$

must be of the form

$$(3) \qquad \rho_1 df_1 + \ldots + \rho_s df_s = 0,$$

where $\rho_1, \ldots, \rho_s$ are undetermined functions of the coordinates of corresponding elements.

We have, by equating the coefficients of dx'_i,

$$-p'_i = \rho_1 \frac{\partial f_1}{\partial x'_i} + \ldots + \rho_s \frac{\partial f_s}{\partial x'_i}.$$

Similarly we obtain other identities by equating the coeffi-

cients of dz', dz, and so on; and we thus have $(2n+2-s)$ equations between the coordinates of corresponding elements when we eliminate the undetermined functions.

If we add to these the s generating equations and eliminate ρ, we shall have $(2n+1)$ equations connecting the coordinates of corresponding elements.

These $(2n+1)$ equations must be equivalent to the system (1). For they are deduced from (1) and the Pfaffian equation (2), which itself follows from (1); they are also unconnected, since they satisfy (2); finally therefore, being $(2n+1)$ in number, unconnected, and following from (1), they are equivalent to (1).

The *generating* equations alone can therefore determine the contact transformation scheme; and it is from this point of view that we shall study them in this chapter.

§ 189. Any s equations connecting the two sets of variables

$$x_1, \ldots, x_n, z \text{ and } x_1', \ldots, x_n', z'$$

may in general be taken as generating equations. They must however satisfy two conditions, viz. firstly the s equations, together with the $(2n+1-s)$ derived equations, must be such that we can by means of them express $x_1', \ldots, x_n', z', p_1', \ldots, p_n'$ in terms of $x_1, \ldots, x_n, z, p_1, \ldots, p_n$; and secondly we must be able to express $x_1, \ldots, x_n, z, p_1, \ldots, p_n$ in terms of

$$x_1', \ldots, x_n', z', p_1', \ldots, p_n'.$$

These two conditions are however equivalent; for suppose that from the assumed system we deduce

$$(1) \quad z' = Z, \ x_i' = X_i, \ p_i' = P_i, \qquad (i = 1, \ldots, n),$$

then by the method of formation of the system we must have

$$dZ - \sum^{i=n} P_i dX_i = \rho\,(dz - \sum^{i=n} p_i dx_i).$$

Now ρ cannot be zero: for if it were the equation (2) of § 188 could not lead to (1), but must lead to exactly $(n+1-s)$ equations connecting $x_1', \ldots, x_n', z', p_1', \ldots, p_n'$. Since then ρ is not zero, the functions $Z, X_1, \ldots, X_n, P_1, \ldots, P_n$ must (by § 178) be unconnected; and therefore

$$x_1, \ldots, x_n, z, p_1, \ldots, p_n$$

can be expressed in terms of $x_1', \ldots, x_n', z', p_1', \ldots, p_n'$.

§ 190. If we are given s equations which cannot be used as generating equations of a contact transformation scheme, what special property will distinguish these equations? We shall call such a system of equations *special* equations. From s special equations we can, as in the general case, deduce $(2n+1-s)$ other equations; and these equations will be unconnected, and will satisfy the Pfaffian equation

$$dz' - \sum^{i=n} p'_i\, dx'_i = \rho\,(dz - \sum^{i=n} p_i\, dx_i).$$

If in the special equations we keep $x'_1, \ldots, x'_n, z'$ all fixed, that is, if we regard this set of variables as parameters, the special equations together with the derived equations will form a system satisfying Pfaff's equation

$$(1) \qquad dz - \sum^{i=n} p_i\, dx_i = 0.$$

If we now consider how the $(2n+1-s)$ derived equations were obtained, we shall see that we can eliminate $p'_1, \ldots, p'_n$, and obtain exactly $(n+1-s)$ derived equations not involving these quantities; these taken with the s special equations will satisfy Pfaff's equation (1).

From that property of the equations, which makes them incapable of being taken as generating equations, we see that we must be able to eliminate the coordinates

$$x'_1, \ldots, x'_n, z', p'_1, \ldots, p'_n,$$

and so obtain at least one equation connecting

$$x_1, \ldots, x_n, z, p_1, \ldots, p_n.$$

Suppose we thus obtain r equations

$$(2) \quad \phi_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0, \qquad (i = 1, \ldots, r);$$

then for all values of the parameters $x'_1, \ldots, x'_n, z'$ the equations

$$f_i(x_1, \ldots, x_n, z, x'_1, \ldots, x'_n, z') = 0, \qquad (i = 1, \ldots, s)$$

will be the generating equations (and therefore, in Lie's sense, an integral) of an M_n satisfying the system of differential equations (2) (see § 155).

§ 191. We shall now limit ourselves to the case of $n = 3$ which offers the most interesting geometrical applications of contact transformation theory.

We take x, y, z as the coordinates of a point, and x, y, z, p, q, as the coordinates of an element in one space; and we take x', y', z', p', q', to be the coordinates of the corresponding element in the other space.

There may now be 1, 2, or 3 generating equations.

We first take the case where there is only one generating equation.

Let this equation be

$$\phi\,(x, y, z, x', y', z') = 0.$$

We now know that the Pfaffian equation

$$dz' - p'dx' - q'dy' - \rho\,(dz - p\,dx - q\,dy) = 0$$

is of the form $d\phi = 0$; and therefore we get as the equations defining the contact transformation scheme

$$(1) \qquad p\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial x} = 0, \quad q\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial y} = 0,$$

$$p'\Big(\frac{\partial\phi}{\partial z'} + \frac{\partial\phi}{\partial x'}\Big) = 0, \quad q'\Big(\frac{\partial\phi}{\partial z'} + \frac{\partial\phi}{\partial y'}\Big) = 0, \quad \phi = 0.$$

The condition, that the coordinates of one element can by aid of these equations be expressed in terms of the corresponding element, shows that the three equations

$$(2) \quad p + \Big(\frac{\partial\phi}{\partial x} \div \frac{\partial\phi}{\partial z}\Big) = 0, \quad q + \Big(\frac{\partial\phi}{\partial y} \div \frac{\partial\phi}{\partial z}\Big) = 0, \quad \phi = 0$$

must be unconnected in the variables x', y', z'.

It follows therefore, after some simple algebraic reduction, that the determinantal equation

$$(3) \qquad \begin{vmatrix} \dfrac{\partial^2\phi}{\partial x\,\partial x'}, & \dfrac{\partial^2\phi}{\partial x\,\partial y'}, & \dfrac{\partial^2\phi}{\partial x\,\partial z'}, & \dfrac{\partial\phi}{\partial x} \\ \dfrac{\partial^2\phi}{\partial y\,\partial x'}, & \dfrac{\partial^2\phi}{\partial y\,\partial y'}, & \dfrac{\partial^2\phi}{\partial y\,\partial z'}, & \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial^2\phi}{\partial z\,\partial x'}, & \dfrac{\partial^2\phi}{\partial z\,\partial y'}, & \dfrac{\partial^2\phi}{\partial z\,\partial z'}, & \dfrac{\partial\phi}{\partial z} \\ \dfrac{\partial\phi}{\partial x'}, & \dfrac{\partial\phi}{\partial y'}, & \dfrac{\partial\phi}{\partial z'}, & 0 \end{vmatrix} = 0$$

must be unconnected with $\phi = 0$.

We could not therefore take as a generating equation

$$xx' + yy' + zz' = 0,$$

for the determinantal equation, formed from it, would be connected with it, as may be easily verified.

This is an example of a *special* equation; the M_2 defined by the equations

$$xx'+yy'+zz' = 0, \quad pz'+x' = 0, \quad qz'+y' = 0$$

must therefore be such that we can eliminate x', y', z' from these equations; if we do so, we obtain the equation

$$px+qy-z = 0,$$

which is satisfied by

$$xx'+yy'+zz' = 0,$$

for all values of the parameters x', y', z'.

From the symmetry of the equation (3) in the two sets of variables x, y, z and x', y', z', we verify the theorem of § 189 as to the equivalence of the two limiting conditions, imposed on the general arbitrariness of the generating equations.

§ 192. If $\phi = 0$ is a generating equation of a contact transformation scheme, the determinantal equation (3) of article § 191 will be unconnected with $\phi = 0$. If then we eliminate x', y', z' between the equations (2) and (3), we shall obtain an equation connecting x, y, z, p, q. Elements satisfying this equation will be called *special* elements.

The equations (1) of § 191 will in general determine one definite element x', y', z', p', q' to correspond to each element x, y, z, p, q. If, however, x, y, z, p, q are the coordinates of a special element it will not have a definite element corresponding to it, but an infinity of elements. Similarly, we shall have special elements in space x', y', z'.

A particular system of special elements may be obtained thus: eliminate x', y', z' from the equations

$$\phi = 0, \quad \frac{\partial\phi}{\partial x'} = 0, \quad \frac{\partial\phi}{\partial y'} = 0, \quad \frac{\partial\phi}{\partial z'} = 0;$$

the resulting equation in x, y, z is known as the *special envelope* of

$$\phi(x, y, z, x', y', z') = 0,$$

x', y', z' being regarded as parameters.

The element consisting of a point on the special envelope together with the tangent plane at the point will be a *special* element; to this special element there will correspond an ∞^2

of elements, consisting of the point x', y', z' together with the ∞^2 of planes through this point.

§ 193. There are three different classes of element manifolds in three-dimensional space. There is, firstly, the manifold M_2 generated from one equation only; such a manifold we shall call a surface M_2.

Let $$f(x, y, z) = 0,\quad p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} = 0,\quad q\frac{\partial f}{\partial z} + \frac{\partial f}{\partial y} = 0$$

be the Pfaffian system of a surface M_2; and let

$$\phi(x, y, z, x', y', z') = 0$$

be the equation which generates the contact transformation scheme.

The generating equation (or it may be equations) of the M_2 which corresponds in the space x', y', z' is that one obtained by eliminating x, y, z from the four equations

$$\frac{\partial f}{\partial x} \div \frac{\partial f}{\partial z} = \frac{\partial \phi}{\partial x} \div \frac{\partial \phi}{\partial z},\quad \frac{\partial f}{\partial y} \div \frac{\partial f}{\partial z} = \frac{\partial \phi}{\partial y} \div \frac{\partial \phi}{\partial z},\quad f = 0,\quad \phi = 0.$$

If we regard x, y, z as variable parameters connected by the equation $f(x, y, z) = 0$, the generating equation is therefore the envelope of

$$\phi(x, y, z, x', y', z') = 0.$$

The manifold M_2 with two generating equations we call a curve M_2.

Let the Pfaffian system of a curve M_2 be

$$f_1(x, y, z) = 0,\quad f_2(x, y, z) = 0,$$

and the equation obtained by eliminating $\lambda : \mu$ from the equations

$$p\left(\lambda\frac{\partial f_1}{\partial z} + \mu\frac{\partial f_2}{\partial z}\right) + \lambda\frac{\partial f_1}{\partial x} + \mu\frac{\partial f_2}{\partial x} = 0,$$

$$q\left(\lambda\frac{\partial f_1}{\partial z} + \mu\frac{\partial f_2}{\partial z}\right) + \lambda\frac{\partial f_1}{\partial y} + \mu\frac{\partial f_2}{\partial y} = 0;$$

that is, the Pfaffian system

$$f_1 = 0,\quad f_2 = 0,\quad p\frac{\partial(f_1, f_2)}{\partial(y, z)} + q\frac{\partial(f_1, f_2)}{\partial(z, x)} = \frac{\partial(f_1, f_2)}{\partial(x, y)}.$$

The generating equation of the M_2, which corresponds in

space x', y', z', is therefore obtained by eliminating x, y, z from

$$f_1 = 0, \quad f_2 = 0, \quad \frac{\partial(\phi, f_1, f_2)}{\partial(x, y, z)} = 0.$$

This generating equation will be the envelope of

$$\phi(x, y, z, x', y', z'),$$

where the parameters x, y, z are connected by

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0.$$

The manifold M_2, which consists of the fixed point a, b, c with the ∞^2 of planes through it, has as the generating equation of the corresponding M_2 in space x', y', z' the surface

$$\phi(a, b, c, x', y', z') = 0.$$

§ 194. If two surface manifolds have a common element they must touch; if two curve manifolds have a common element they intersect; and if a curve manifold has an element common with a surface M_2 they also touch.

If a point M_2 has an element common with a surface M_2 or a curve M_2, the point must lie on that surface, or on that curve; but two point manifolds cannot have any common element, unless they coincide entirely.

If then in space x, y, z two different M_2's have a common element, the M_2's in space x', y', z' which correspond to these will also have in general a common element; the exceptional case is when the first common element is a special one.

Thus, if the two surfaces

$$\phi(x, y, z, a_1, b_1, c_1) = 0 \quad \text{and} \quad \phi(x, y, z, a_2, b_2, c_2) = 0$$

touch, the common element must be a special one for the contact transformation with the generating equation

$$\phi(x, y, z, x', y', z') = 0.$$

For otherwise the M_2 consisting of the point a_1, b_1, c_1 with the ∞^2 of planes through this point would have a definite element common with the point M_2 whose coordinates are a_2, b_2, c_2, and this is of course impossible.

So if two M_2's have an infinity of common elements, the corresponding surfaces will also generally have an infinity of common elements.

Thus, if two surface M_2's have an infinity of common elements, they must either touch along a common curve; or have a common conical point, and the same tangent cone at

the conical point; if the corresponding M_2's in the other space are also surface manifolds they must also have one of these properties.

Again, if a curve A is traced on a surface B, then if A is transformed to a curve A', and B to a surface B', we must have A' traced on B'; if, however, A is transformed into a surface A' and B into a surface B', the two surfaces A' and B' must either have a common conical point, with a common tangent cone at it, or they must touch along a common curve.

Again, if A and B are two points, then the straight line joining these points will be a curve M_2, with one infinity of elements common to the point manifold A, and another infinity of elements common to the point manifold B; if then this straight line M_2 is transformed to a curve M_2 it will be the curve common to the two surfaces A' and B'; if, however, it is transformed into a surface M_2, it will generally be a surface touching A' along one curve, and B' along another curve.

§ 195. The most interesting example of contact transformation of the first class is obtained by taking the generating equation $\phi = 0$ to be linear both in x', y', z' and in x, y, z, viz.

$$x(a_1 x' + b_1 y' + c_1 z' + d_1) + y(a_2 x' + b_2 y' + c_2 z' + d_2) \\ + z(a_3 x' + b_3 y' + c_3 z' + d_3) + a_4 x' + b_4 y' + c_4 z' + d_4 = 0.$$

We see at once that the only limitation placed on the constants in this equation, in order that $\phi = 0$ may generate a contact transformation, is that the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}$$

should not vanish.

If this condition is satisfied the equation $\phi = 0$ will generate a contact transformation; and, since the determinant does not contain any variables, the contact transformation will be one with respect to which there are no special elements.

Clearly a point in either space will correspond to a plane in the other; and the straight line given by

$$\alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 = 0, \quad \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 = 0$$

will be transformed to an M_2 whose generating equation is the envelope of the plane $\phi = 0$, when we regard x, y, z as the parameters. This envelope is a straight line, and therefore

the contact transformation transforms straight lines into straight lines.

If we take as the generating equation

$$\phi = xx' + yy' - z - z' = 0$$

—a form to which any equation, linear both in x, y, z and x', y', z', can be reduced by a projective point transformation—we have the well-known contact transformation

$$p = x', \quad q = y', \quad p' = x, \quad q' = y, \quad z' = px + qy - z;$$

this is geometrically equivalent to reciprocation with respect to the paraboloid of revolution

$$2z = x^2 + y^2.$$

§ 196. We now proceed to discuss at greater length the second kind of contact transformations, viz. those in which there are two generating equations.

Let these equations be

$$\phi\,(x, y, z, x', y', z') = 0, \quad \psi\,(x, y, z, x', y', z') = 0\,;$$

then, since the equation

$$dz' - p'dx' - q\,dy' - \rho\,(dz - p\,dx - q\,dy) = 0$$

is to be of the form

$$\lambda\,d\phi + \mu\,d\psi = 0,$$

we must have

$$p'\left(\lambda\frac{\partial\phi}{\partial z'} + \mu\frac{\partial\psi}{\partial z'}\right) + \lambda\frac{\partial\phi}{\partial x'} + \mu\frac{\partial\psi}{\partial x'} = 0,$$

$$q'\left(\lambda\frac{\partial\phi}{\partial z'} + \mu\frac{\partial\psi}{\partial z'}\right) + \lambda\frac{\partial\phi}{\partial y'} + \mu\frac{\partial\psi}{\partial y'} = 0,$$

$$p\left(\lambda\frac{\partial\phi}{\partial z} + \mu\frac{\partial\psi}{\partial z}\right) + \lambda\frac{\partial\phi}{\partial x} + \mu\frac{\partial\psi}{\partial x} = 0,$$

$$q\left(\lambda\frac{\partial\phi}{\partial z} + \mu\frac{\partial\psi}{\partial z}\right) + \lambda\frac{\partial\phi}{\partial y} + \mu\frac{\partial\psi}{\partial y} = 0.$$

If we eliminate the undetermined function $\lambda : \mu$ from these equations we shall have three equations; and these, together with the generating equations, determine the contact transformation scheme.

The equations $\phi = 0, \psi = 0$, may be any whatever, provided that the above five equations determine an element of one space in terms of the corresponding element in the other space.

If we take W to denote $\lambda\phi+\mu\psi$, and in differentiating regard λ and μ as mere constants, we may express this limitation by saying that the four equations

$$\phi = 0, \quad \psi = 0, \quad p\frac{\partial W}{\partial z} + \frac{\partial W}{\partial x} = 0, \quad q\frac{\partial W}{\partial z} + \frac{\partial W}{\partial y} = 0$$

in the variables $x', y', z', \lambda:\mu$ are unconnected.

It may be proved without much labour that this condition is equivalent to saying that the determinant

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial x}, & \dfrac{\partial\phi}{\partial y}, & \dfrac{\partial\phi}{\partial z}, & 0, & 0 \\[2ex] \dfrac{\partial\psi}{\partial x}, & \dfrac{\partial\psi}{\partial y}, & \dfrac{\partial\psi}{\partial z}, & 0, & 0 \\[2ex] \dfrac{\partial^2 W}{\partial x\partial x'}, & \dfrac{\partial^2 W}{\partial y\partial x'}, & \dfrac{\partial^2 W}{\partial z\partial x'}, & \dfrac{\partial\psi}{\partial x'}, & \dfrac{\partial\phi}{\partial x'} \\[2ex] \dfrac{\partial^2 W}{\partial x\partial y'}, & \dfrac{\partial^2 W}{\partial y\partial y'}, & \dfrac{\partial^2 W}{\partial z\partial y'}, & \dfrac{\partial\psi}{\partial y'}, & \dfrac{\partial\phi}{\partial y'} \\[2ex] \dfrac{\partial^2 W}{\partial x\partial z'}, & \dfrac{\partial^2 W}{\partial y\partial z'}, & \dfrac{\partial^2 W}{\partial z\partial z'}, & \dfrac{\partial\psi}{\partial z'}, & \dfrac{\partial\phi}{\partial z'} \end{vmatrix}$$

must not vanish by aid of $\phi = 0, \psi = 0$ for all values of $\lambda:\mu$; that is, the determinantal equation must be unconnected with $\phi = 0, \psi = 0$.

If we substitute in this determinant for $x', y', z', \lambda:\mu$ their values in terms of x, y, z, p, q obtained from (1), and equate the result to zero, we shall have the equation satisfied by *special* elements in the space x, y, z.

§ 197. In accordance with § 190, we notice that the limitation placed on the generating equations is that $\phi = 0, \psi = 0$ must not be, for all values of the parameters x', y', z', the integral of any partial differential equation of the first order.

Example. It may at once be verified that we could not take as generating equations

$$axx' + byy' + czz' = 0, \quad xx' + yy' + zz' = 0.$$

If, regarding x', y', z' as parameters, we complete the Pfaffian system of which these are the two generating equations, we have as the third equation

$$(a-b)\,x'y' = (b-c)\,py'z' + (c-a)\,qx'z'.$$

Eliminating the parameters x', y', z' we get as one of the equations of the Pfaffian system

$$z = px + qy;$$

and we see that, according to Lie's definition,

$$axx' + byy' + czz' = 0, \quad xx' + yy' + zz' = 0,$$

is therefore a complete integral of

$$z = px + qy.$$

In this, as in all classes of contact transformations, the general principle holds that two M_2's with a common element are transformed into two M_2's with a common element.

§ 198. Before proceeding to discuss the applications of this class of contact transformations to geometry, we write down some elementary properties of complexes of lines, which will prove useful in the sequel.

We take as the coordinates of a line whose direction cosines are l, m, n, and which passes through the point x', y', z'

$$l, m, n, \alpha, \beta, \gamma,$$

where

$$\alpha = mz' - ny', \quad \beta = nx' - lz', \quad \gamma = ly' - mx'.$$

If the coordinates of a line are connected by the linear relation $\quad \alpha' l + \beta' m + \gamma' n + l'\alpha + m'\beta + n'\gamma = 0,$

where l', m', n', α', β', γ' are *any* given constants, the line is said to belong to a given *linear complex*; l', m', n', α', β', γ' are said to be the coordinates of the complex. If the coordinates of the complex are connected by the equation

$$l'\alpha' + m'\beta' + n'\gamma' = 0,$$

then the coordinates of the complex are the coordinates of a line, and the complex consists of straight lines intersecting a given line.

We may take l', m', n' to be forces along the axes of coordinates; and α', β', γ' to be couples whose axes coincide with the axes of coordinates. If a rigid body is rotated about the line l, m, n, α, β, γ through a small angle dt, it has linear displacements αdt, βdt, γdt along the axes, and rotations $l dt$, $m dt$, $n dt$ about them. The work done by the given forces and couples is then

$$(l'\alpha + m'\beta + n'\gamma + l\alpha' + m\beta' + n\gamma')\, dt;$$

and therefore, if a body is rotated about any line of the complex, the given system of forces do no work on it.

These statical considerations enable us to simplify the equation of a linear complex; for, if we take the wrench equivalent to the given system of forces and couples, we know that it acts along a fixed line, which we now call the axis of the complex; let k be the ratio of the couple to the force in the wrench, and let us take the axis of the wrench as the axis of z. We now have

$$l' = 0,\ m' = 0,\ \alpha' = 0,\ \beta' = 0,\ \gamma' = kn',$$

and therefore, if a line be such that the wrench does no work on a rigid body rotating about it, its coordinates must satisfy the equation

$$\gamma + kn = 0;$$

this therefore is a form to which any given linear complex can be reduced.

An infinity of lines can be drawn through any point x', y', z' which shall belong to the complex $\gamma + kn = 0$; these lines all lie on the plane $yx' - xy' + k(z' - z) = 0$, which is called a null plane of the complex. Through every point a null plane can be drawn.

Any two lines, whose coordinates are

$$l,\ m,\ n,\ \alpha,\ \beta,\ \gamma,$$

$$l,\ m,\ \frac{-\gamma}{k},\ \alpha,\ \beta,\ -kn,$$

are said to be conjugate to one another with respect to the complex.

If x', y', z' lies on any straight line the conjugate line lies on the null plane of x', y', z'; and the null planes of two points intersect on the line conjugate to the join of the two points.

If the coordinates of two complexes

$$l_1,\ m_1,\ n_1,\ \alpha_1,\ \beta_1,\ \gamma_1,$$

$$l_2,\ m_2,\ n_2,\ \alpha_2,\ \beta_2,\ \gamma_2$$

are connected by the equation

$$l_1 a_2 + l_2 a_1 + m_1\beta_2 + m_2\beta_1 + n_1\gamma_2 + n_2\gamma_1 = 0,$$

they are said to be in involution. The statical interpretation is that a wrench along the axis of one complex does no work in a rigid body, which is moved along the screw of the other.

The two complexes, whose coordinates are respectively

$$l', m', n', a', \beta', \gamma',$$

$$l', m', \frac{\gamma'}{k}, a', \beta', -kn',$$

are said to be conjugate with respect to the complex

$$\gamma + kn = 0.$$

If a line belongs to any complex, its conjugate line belongs to the conjugate complex.

If two lines intersect, their conjugate lines also intersect.

A line coincides with its conjugate, if, and only if, it belongs to the complex, with respect to which the lines are conjugate.

§ 199. Let us now take as our generating equations for the contact transformation the bilinear equations

$$x(a_1x' + b_1y' + c_1z' + d_1) + y(a_2x' + b_2y' + \ldots) \\ + z(a_3x' + b_3y' + \ldots) + a_4x' + b_4y' + \ldots = 0,$$

$$x(a_1x' + \beta_1y' + \gamma_1z' + \delta_1) + y(a_2x' + \beta_2y' + \ldots) \\ + z(a_3x' + \beta_3y' + \ldots) + a_4x' + \beta_4y' + \ldots = 0.$$

If we keep x', y', z' fixed, these are the equations of two planes; in order to simplify the form of the equations by a projective transformation, we consider the positions of the point x', y', z', which will cause these planes to be coincident.

For the coincidence of the planes we must have

$$(1)\quad \frac{a_1x' + b_1y' + c_1z' + d_1}{a_1x' + \beta_1y' + \gamma_1z' + \delta_1} = \frac{a_2x' + \ldots}{a_2x' + \ldots} = \frac{a_3x' + \ldots}{a_3x' + \ldots} = \frac{a_4x' + \ldots}{a_4x' + \ldots};$$

equating these equal fractions to λ, and eliminating x', y', z', we have

$$\begin{vmatrix} a_1 - \lambda a_1, & b_1 - \lambda\beta_1, & c_1 - \lambda\gamma_1, & d_1 - \lambda\delta_1 \\ a_2 - \lambda a_2, & b_2 - \lambda\beta_2, & c_2 - \lambda\gamma_2, & d_2 - \lambda\delta_2 \\ a_3 - \lambda a_3, & b_3 - \lambda\beta_3, & c_3 - \lambda\gamma_3, & d_3 - \lambda\delta_3 \\ a_4 - \lambda a_4, & b_4 - \lambda\beta_4, & c_4 - \lambda\gamma_4, & d_4 - \lambda\delta_4 \end{vmatrix} = 0.$$

There are in general, therefore, four positions of the point x', y', z', for which the generating equations will represent the same plane.

We first consider the case where the four points lie on the same plane; and, by a projective transformation, we may take this plane to be the plane at infinity.

The points therefore which give coincident planes must satisfy the equations (1), when in these we put

$$d_1 = 0,\ \delta_1 = 0,\ d_2 = 0,\ \delta_2 = 0, \ldots;$$

and therefore all three-rowed determinants must vanish in the matrix

$$\left\| \begin{matrix} a_1 - \lambda\alpha_1, & a_2 - \lambda\alpha_2, & a_3 - \lambda\alpha_3, & a_4 - \lambda\alpha_4 \\ b_1 - \lambda\beta_1, & b_2 - \lambda\beta_2, & b_3 - \lambda\beta_3, & b_4 - \lambda\beta_4 \\ c_1 - \lambda\gamma_1, & c_2 - \lambda\gamma_2, & c_3 - \lambda\gamma_3, & c_4 - \lambda\gamma_4 \end{matrix} \right\|.$$

Now these are cubic equations in λ, and by hypothesis they are satisfied for four values of λ; they must therefore be identically true for all values of λ.

The deduction of the necessary relations between the constants, involved in these identities, is made easy by a geometrical representation.

We take A_1 to be a point whose coordinates are a_1, b_1, c_1, B_1 to be the point whose coordinates are $\alpha_1, \beta_1, \gamma_1$, and so on.

Taking $\lambda = 0$ we see that A_1, A_2, A_3 are three collinear points; taking λ to be infinite we see that B_1, B_2, B_3 are collinear. It now follows, from the given identities, that any three points which divide the three lines A_1B_1, A_2B_2, A_3B_3, in the same ratio are themselves collinear. These three lines must therefore be generators of a paraboloid of which two generators (of the opposite system) are $A_1A_2A_3$ and $B_1B_2B_3$. It follows that A_3 divides A_1A_2 in the same ratio that B_3 divides B_1B_2.

Similarly we see that A_1, A_2, A_3, A_4 are four collinear points dividing their line in the same ratios that B_1, B_2, B_3, B_4 divide their line.

§ 200. If we now take

$$X' = a_1x' + b_1y' + c_1z', \quad Y' = a_2x' + b_2y' + c_2z',$$
$$Z' = \alpha_1x' + \beta_1y' + \gamma_1z', \quad W' = \alpha_2x' + \beta_2y' + \gamma_2z',$$

we see that the generating equations must be of the form

$$x(X' + d_1) + y(Y' + d_2) + z(pX' + qY' + d_3) + p'X' + q'Y' + d_4 = 0,$$
$$x(Z' + \delta_1) + y(W' + \delta_2) + z(pZ' + qW' + \delta_3) + p'Z' + q'W' + \delta_4 = 0,$$

where p, q, p', q' are some constants.

We further simplify these equations by taking

$$X=\frac{x+pz+p'}{y+qr+q'},\quad Z=\frac{d_1x+d_2y+d_3z+d_4}{y+qz+q'},$$

$$W=\frac{\delta_1x+\delta_2y+\delta_3z+\delta_4}{y+qz+q'},$$

when we have as generating equations

$$XX'+Y'+Z=0,\quad XZ'+W'+W=0,$$

where X', Y', Z', W' are connected by an identity of the form

$$aX'+bY'+cZ'+dW'\equiv 0.$$

If finally we take new sets of variables x, y, z and x', y', z', given by

$$x=X,\quad z=aZ+cW,\quad y=-bZ-dW,$$

$$x'+iy'=aY'+cW',\quad x'-iy'=bX'+dZ',\quad z'=bY'+dW',$$

where i is the symbol $\sqrt{-1}$, the generating equations reduce to

$$xz'+z+x'+iy'=0,\quad x(x'-iy')-z'-y=0.$$

To sum up: when the four points in space x', y', z' which make the generating equations coincident are coplanar, the generating equations can by a projective transformation be thrown into the standard form

$$xz'+z+x'+iy'=0,\quad x(x'-iy')-z'-y=0.$$

In this standard form we now see that every point has this property which lies on the intersection of the cone

$$x'^2+y'^2+z'^2=0$$

with the plane at infinity; that is, any point on the absolute circle at infinity has the property of making the generating equations coincident.

§ 201. We must now study the contact transformation with these generating equations

$$(1)\quad x'+iy'+xz'+z=0,\quad x(x'-iy')-y-z'=0.$$

It is to be noticed that, as the equations are not symmetrical in the coordinates of the two spaces, the relation between the corresponding elements in the spaces will not be symmetrical.

In addition to (1) we have for determining the transformation

$$p'(x-q)+1+qx=0, \quad q'(x-q)+i(1-qx)=0,$$
$$p+z'+q(x'-iy')=0;$$

and we see that each element in space x', y', z' can be uniquely determined in terms of the corresponding element in space x, y, z.

If, however, we wish to express x, y, z, p, q in terms of x', y', z', p', q', we have, to determine x and q, the equations

$$qx=\frac{p'+iq'}{p'-iq'}, \quad q-x=\frac{2}{p'-iq'};$$

and therefore two different elements in space x, y, z will have the same correspondent in space x', y', z'.

Such a pair of elements in space x, y, z we shall call *conjugate* elements; it may easily be proved that the contact transformation

$$x'=-q, \quad y'=p, \quad p'=y, \quad q'=-x, \quad z'=z-px-qy$$

will transform any element to its conjugate element.

Example. Prove that this contact transformation is the result of first reciprocating with respect to $xy=2z$, and then reflecting the surface with respect to the axis of y.

Reciprocation is equivalent to taking as our generating equation $$xy'+yx'-z-z'=0;$$
and therefore

$$x'=q, \quad y'=p, \quad z'=px+py-z, \quad p'=y, \quad q'=x.$$

If we now reflect with respect to the axis of y, we have

$$z''=-z', \quad x''=-x', \quad y''=y';$$

and completing the contact transformation, generated by these three equations, we have

$$p''=p', \quad q''=-q',$$

so that

$$z''=z-px-qy, \quad x''=-q, \quad y''=p, \quad p''=y, \quad q''=-x.$$

Example. Prove that if the element x, y, z, p, q is rotated 90° round the axis of z, in the positive direction, and the conjugate element x', y', z', p', q' is reflected in the plane $z=0$,

the two resulting elements will be reciprocal with respect to $x^2+y^2=2z$, that is, will be connected by the equations

$$z+z'=px+qy, \quad x'=p, \quad y'=q, \quad x=p', \quad y=q'.$$

§ 202. To the point x', y', z' there will correspond in space x, y, z the straight line given by the generating equations when we regard x', y', z' as fixed. The only exceptional case is when x', y', z' lies in the absolute circle in its space, and then we have as its correspondent a plane in the other space.

The six coordinates of the straight line corresponding to x', y', z' are given by

$$\frac{l}{1}=\frac{m}{x'-iy'}=\frac{n}{-z'}=\frac{a}{-(x'^2+y'^2+z'^2)}=\frac{\beta}{x'+iy'}=\frac{\gamma}{-z'};$$

all of these lines are therefore lines of the linear complex $\gamma=n$.

To the point x, y, z there will correspond in space x', y', z' the straight line whose coordinates are given by

$$\frac{l}{i(x^2-1)}=\frac{m}{x^2+1}=\frac{n}{-2ix}$$
$$=\frac{a}{\dfrac{ixz}{x^2-1}-y}=\frac{\beta}{\dfrac{xz}{x^2-1}-iy}=\frac{\gamma}{\dfrac{iz}{x^2-1}-xy}.$$

This straight line is such that

$$l^2+m^2+n^2=0,$$

and therefore to x, y, z there corresponds in the other space a minimum straight line.

It will be noticed that, in order to find what corresponds to a point M_2, it is only necessary to make use of the coordinates of the point and the generating equations. In order to find what corresponds to the surface M_2 given by

$$lx+my+nz+k=0,$$

we must form the other Pfaffian equations of this M_2 viz.

$$l+np=0, \quad m+nq=0.$$

From the equations of the contact transformation we now have

$$(1) \qquad l+m(x'-iy')-nz'=0.$$

Eliminating x and y from the generating equations and the

equation of the given plane, we see that (on account of (1)) z also disappears, and we get

$$n(x'+iy')+mz'-k=0.$$

The plane therefore has as its correspondent the minimum line

$$n(x'+iy')+mz'-k=0, \quad l+m(x'-iy')-nz'=0;$$

that is, has the same correspondent as the point

$$x=\frac{m}{n}, \quad y=\frac{-l}{n}, \quad z=\frac{-k}{n}.$$

§ 203. We next find what will correspond to the straight line

$$(1) \qquad \alpha=mz-ny, \quad \beta=nx-lz, \quad \gamma=ly-mx,$$

of which the coordinates are $l, m, n, \alpha, \beta, \gamma$.

Eliminating x, y, z from two of these equations (there are of course only two unconnected ones) and the generating equations, we clearly get the generating equation of the M_2 we require; it is

$$(2) \quad l(x'^2+y'^2+z'^2)-\beta(x'-iy')-m(x'+iy')+(n+\gamma)z'-\alpha=0.$$

To find the minimum straight line, which corresponds to any point on the given line $l, m, n, \alpha, \beta, \gamma$, we must substitute in the generating equations for y and z their values in terms of x; we get

$$x(lz'+n)=\beta-l(x'+iy),$$
$$x\big(l(x'-iy')-m\big)=\gamma+lz'.$$

Eliminating x from these two equations, we get the equation of the sphere which corresponds to the given straight line; and one set of generators on this sphere consists of the minimum lines which correspond to points on the given line.

Writing the equation of the sphere in the form

$$(3) \quad x'^2+y'^2+z'^2+2gx'+2fy'+2hz'+c=0,$$

and comparing with (2), we do not get unique values for the coordinates of the straight line in terms of the coordinates of the sphere. If we take r to be the radius of the sphere (that is, $\sqrt{f^2+g^2+h^2-c}$ taken positively), we see that there are two straight lines in space x, y, z to each of which the same sphere (3) will correspond.

These lines are respectively

$$\frac{l}{1}=\frac{m}{-g+if}=\frac{n}{h-r}=\frac{\alpha}{-c}=\frac{\beta}{-g-if}=\frac{\gamma}{h+r},$$

which we call the *positive correspondent* of the sphere, and

$$\frac{l}{1}=\frac{m}{-g+if}=\frac{n}{h+r}=\frac{a}{-c}=\frac{\beta}{-g-if}=\frac{\gamma}{h-r},$$

which we call the *negative correspondent.*

These two lines are conjugate with respect to the linear complex $\gamma = n$.

When $r = 0$, the sphere degenerates into a cone; and any plane through the vertex is a tangent plane to the cone (though of course an infinity of planes through the vertex are tangent planes in a more special sense).

The two lines, the positive and negative correspondents of the degenerate sphere, now coincide; and therefore belong to the linear complex $\gamma = n$. This is another way of obtaining the fundamental theorem, that a point in space x', y', z' has as its correspondent in the other space a straight line of the linear complex $\gamma = n$.

By allowing f, g, h, c to increase indefinitely, without altering their mutual ratios, we see that to the plane

$$2gx'+2fy'+2hz'+c=0,$$

there are two correspondents in space x, y, z, viz. the positive correspondent

$$l=0,\ \frac{m}{-g+if}=\frac{n}{h-\sqrt{h^2+g^2+f^2}}$$
$$=\frac{a}{-c}=\frac{\beta}{-g-if}=\frac{\gamma}{h+\sqrt{h^2+g^2+f^2}},$$

and the negative correspondent obtained by changing the sign of the surd.

The straight lines therefore, which are perpendicular to the axis of x, are not transformed into spheres, but into planes.

§ 204. Suppose now that we have the two spheres

$$x'^2+y'^2+z'^2+2g_1x'+2f_1y'+2h_1z'+c_1=0,$$
$$x'^2+y'^2+z'^2+2g_2x'+2f_2y'+2h_2z'+c_2=0,$$

then, if

$$l_1, m_1, n_1, a_1, \beta_1, \gamma_1,$$
$$l_2, m_2, n_2, a_2, \beta_2, \gamma_2$$

are the line coordinates of their positive correspondents, we have

$$l_1a_2+l_2a_1=-c_1-c_2,\ m_1\beta_2+m_2\beta_1=2g_1g_2+2f_1f_2,$$
$$n_1\gamma_2+n_2\gamma_1=2h_1h_2-2r_1r_2,$$

so that if the positive correspondents intersect,

$$2g_1g_2 + 2f_1f_2 + 2h_1h_2 = 2r_1r_2 + c_1 + c_2;$$

that is, the two spheres touch internally.

If the positive correspondents intersect so do the negative; for a positive and negative correspondent are conjugate to the linear complex $\gamma = n$.

If then two spheres touch internally the positive correspondent of the first intersects the positive correspondent of the second; and the negative correspondents also intersect.

The two straight lines, the positive and negative correspondents of a sphere, cannot intersect unless the sphere degenerates into a point sphere; for conjugate lines, with respect to a linear complex, can only intersect when the lines belong to the complex; that is, when $\gamma = n$, and therefore $r = 0$.

If the first positive correspondent intersects the second negative correspondent, then the second positive correspondent intersects the first negative correspondent, and the spheres have external contact.

§ 205. If we are given a line whose six coordinates are

$$l, m, n, \alpha, \beta, \gamma,$$

how are we to decide whether it is a positive or a negative correspondent to the sphere to which it corresponds—for we know there is only one such sphere?

We always suppose the radius of the sphere to be positive, and therefore by the formula

$$2r = \pm \frac{\gamma - n}{l},$$

taking, as we may, l to be positive, we know that the line is a positive correspondent if $\gamma > n$, and a negative if $\gamma < n$.

If then we are given two interesting lines, there is no ambiguity as to whether the corresponding spheres intersect externally or internally; the question is settled by the positions of the line with regard to the axes of coordinates.

If we neglected this consideration we should arrive at paradoxical results by this method of contact transformation. Thus, if we are given two intersecting straight lines A, B, we know that, if any other two straight lines C, D intersect them both, then C, D must themselves intersect. It would therefore appear to follow, from the theory of contact transformation explained, that if two spheres touch one another, then any other pair of spheres, which touch both of the first pair, must

also touch one another, a result which is obviously absurd. To see where the error has arisen in the application of the contact principle, suppose that the first two spheres touch externally; then A and B must be taken to be, one a positive, and the other a negative correspondent of its sphere. We suppose C to be a positive correspondent to its sphere C', A a positive correspondent to its sphere A', and B a negative correspondent to B'; we now have C' touching A' internally and B' externally; and the only way this could happen would be by C' touching the two spheres, at their common point of contact. Similarly D' must touch at this point; and therefore C' and D' do touch one another, but they are not *any* spheres touching both A' and B'.

§ 206. The cyclide of Dupin is the envelope of a sphere which touches three given spheres (Salmon, *Geometry of Three Dimensions*, p. 535), there being four distinct cyclides, corresponding to the different kinds of contact of the variable sphere with the three given spheres A, B, C.

The four cases are when the variable sphere touches, (1) A, B, C all externally or all internally; (2) B, C externally and A internally or B, C internally and A externally; (3) C, A externally and B internally, or C, A, internally and B externally; (4) A, B externally and C internally or A, B internally and C externally.

We shall only consider the first of these cyclides; taking a, b, c, d to be the positive and $-a$, $-b$, $-c$, $-d$ to be the negative correspondents of A, B, C, D we see that, either d intersects a, b, c, or else it intersects the three negative correspondents $-a$, $-b$, $-c$; in either case it generates a surface of the second degree.

A cyclide of Dupin in space x', y', z' therefore generally corresponds to a quadric in space x, y, z. If we take any generator of this quadric and regard it as the generating curve of a curve M_2 in space x, y, z, its correspondent in the other space will be a sphere touching the cyclide along a curve. This curve must be a line of curvature on the cyclide; for the normals to the sphere along this curve intersect, and therefore the normals to the cyclide along this curve intersect.

If, however, instead of regarding the generator of the quadric as a curve M_2 of x', y', z', we regard it as an M_1 of elements of the quadric; that is, if we take the single infinity of elements, consisting of the points of the generator and the tangent planes at these points to the quadric, then the corresponding M_1

in space x', y', z' is the line of curvature, with the tangent planes at each point of it to the cyclide.

§ 207. Any surface in space x, y, z has at every point on it two inflectional tangents. The surface therefore which corresponds in space x', y', z' will have, as corresponding to these two inflectional tangents, two spheres each having contact with the surface at two consecutive points; that is, the correspondents of the inflectional tangents will be the two spheres whose radii are the principal radii of curvature (Salmon, *ibid.*, p. 264).

It will be noticed that any straight line drawn through a point on a surface, and in the tangent plane, will be transformed into a sphere touching the corresponding surface. The peculiar property, however, of an inflectional tangent is that it is a straight line through two consecutive points of a surface, and also in the two consecutive tangent planes at these points. It is therefore transformed into a sphere having two consecutive elements common with the new surface; that is, it is a sphere whose radius is equal to one of the principal radii of curvature.

By this contact transformation therefore the curves, whose tangents are the inflectional tangents to the surface at the point, are transformed so as to become the lines of curvature on the surface in space x', y', z'.

If a surface has any straight line altogether contained in it the corresponding surface will have a line of curvature, with the same radius and centre of curvature all along this line.

§ 208. In general a quadric in space x, y, z is transformed into a cyclide; but we shall now see that some quadrics are transformed into straight lines in space x', y', z'.

Let $\quad a = mz'-ny', \quad \beta = nx'-lz', \quad \gamma = ly'-mx'$

be a straight line in space x', y', z'; from the generating equations we obtain, by eliminating x', y', z',

$$x\left((ai+\beta)x-ny+(mi-l)z-2\gamma i\right) = (l+mi)y+nz+ai-\beta.$$

This quadric therefore, instead of having a cyclide corresponding to it in space x', y', z', has the line whose coordinates are

$$l, m, n, a, \beta, \gamma.$$

It may be verified without difficulty that one system of generators of this quadric belongs to the complex $l = 0$, and the other to the complex $\gamma = n$.

§ 209. If we have a system of concentric spheres in space x', y', z', viz.

$$x'^2+y'^2+z'^2+2gx'+2fy'+2hz'+c=0,$$

where c varies, the corresponding system of manifolds in space x, y, z will be straight lines satisfying the three linear complexes

$$\frac{l}{1}=\frac{m}{-g+if}=\frac{n+\gamma}{2h}=\frac{\beta}{-g-if}.$$

Two different manifolds will correspond to a given sphere of radius r; there will be the positive correspondent obtained by making the coordinates of the straight line also satisfy the linear complex

$$2rl=\gamma-n,$$

and the negative by making the coordinates satisfy the complex

$$2rl=n-\gamma.$$

All these lines are generators of the same system on the hyperboloid

$$(1) \qquad (if-g)\,x^2-xy+2hx-z+if+g=0.$$

The generators of the other (the second) system on (1) are

$$x=t, \quad z+ty=if+g+2ht+(if-g)\,t^2;$$

the six coordinates of any one of these generators are

$$\frac{l}{0}=\frac{m}{1}=\frac{n}{-t}=\frac{a}{if+g+2ht+(if-g)\,t^2}=\frac{\beta}{-t^2}=\frac{\gamma}{-t}.$$

Since $l=0$, to each of these generators there will correspond in space x', y', z' a plane touching all the concentric spheres; these planes must therefore be tangent planes to the asymptotic cone

$$(x'+g)^2+(y'+f)^2+(z'+h)^2=0;$$

this result may be at once directly verified.

It may be noticed that all generators of the second system belong to both the linear complexes

$$l=0 \text{ and } \gamma=n.$$

The hyperboloid (1) is given when we are given a generator of its first system; one such hyperboloid can be described through any straight line. We see therefore how to construct the system of lines which will be transformed into concentric spheres; describe an hyperboloid of the form (1) through any line; then the lines, which will be transformed to concentric spheres, are the infinity of generators

of the same system as the given line. In particular that generator, which belongs to the linear complex $\gamma = n$, will correspond to the centre of the given system of spheres.

§ 210. If a quadric is such that all generators of one system belong to the linear complex $\gamma = n$, then its correspondent in space x', y', z', instead of being a cyclide, is a circle. For we have, in space x, y, z, a system of generators intersecting two fixed generators, and belonging to the complex $\gamma = n$; in the corresponding figure therefore we must have a system of points common to two spheres, that is, a circle.

§ 211. We now pass on to consider the more general case of the two bilinear generating equations, when the four points in space x', y', z', for which the generating equations become coincident, are not coplanar. We take these four points as the vertices of a tetrahedron; and we do not consider the special cases which might arise, owing to two or more of these vertices coinciding. We choose our coordinate axes so that this tetrahedron has for its vertices the points

$$(0, 0, 0), \quad (\infty, 0, 0), \quad (0, \infty, 0), \quad (0, 0, \infty);$$

we thus have from the definition of the tetrahedron (employing the same notation as in § 199)

$$\frac{a_1}{a_1} = \frac{a_2}{a_2} = \frac{a_3}{a_3} = \frac{a_4}{a_4} = \lambda_1, \quad \frac{\beta_1}{b_1} = \frac{\beta_2}{b_2} = \frac{\beta_3}{b_3} = \frac{\beta_4}{b_4} = \lambda_2,$$

$$\frac{\gamma_1}{c_1} = \frac{\gamma_2}{c_2} = \frac{\gamma_3}{c_3} = \frac{\gamma_4}{c_4} = \lambda_3, \quad \frac{\delta_1}{d_1} = \frac{\delta_2}{d_2} = \frac{\delta_3}{d_3} = \frac{\delta_4}{d_4} = \lambda_4.$$

We then take

$$X = \frac{a_1x + a_2y + a_3z + a_4}{d_1x + d_2y + d_3z + d_4}, \quad Y = \frac{b_1x + b_2y + b_3z + b_4}{d_1x + d_2y + d_3z + d_4},$$

$$Z = \frac{c_1x + c_2y + c_3z + c_4}{d_1x + d_2y + d_3z + d_4},$$

and thus see that by projective transformation the generating equations may be thrown into the forms

$$axx' + byy' + czz' + d = 0,$$

$$xx' + yy' + zz' + 1 = 0.$$

If we keep x', y', z' fixed, these are the equations of two planes, and therefore to a point x', y', z' there corresponds

a straight line in space x, y, z. The six coordinates of this line satisfy the equation

$$\frac{l\alpha}{(b-c)(a-d)} = \frac{m\beta}{(c-a)(b-d)} = \frac{n\gamma}{(a-b)(c-d)};$$

that is, the line belongs to a complex of the second degree.

It can be at once verified that every straight line of this complex is divided in a constant anharmonic ratio by the coordinate planes and the plane at infinity; on account of this property the complex is called a *tetrahedral complex*.

We may look on the generating equations as the polar planes of x', y', z', with respect to two quadrics, which do not touch; the quadrics are referred to their common self-conjugate tetrahedron, viz. the coordinate planes and the plane at infinity, and the polar planes intersect in a line of a tetrahedral complex of this tetrahedron.

In order to complete the contact transformation we must add to the generating equations the three equations obtained by eliminating λ from

$$-p = \frac{(\lambda+a)x'}{(\lambda+c)z'}, \quad -q = \frac{(\lambda+b)y'}{(\lambda+c)z'},$$

$$-p' = \frac{(\lambda+a)x}{(\lambda+c)z}, \quad -q' = \frac{(\lambda+b)y}{(\lambda+c)z},$$

that is,

$$p(b-c)z'y' + q(c-a)z'x' - (a-b)x'y' = 0,$$

$$pz'x - p'zx' = 0, \quad qz'y - q'zy' = 0.$$

The equation $p'(b-c)zy + q'(c-a)z'x' - (a-b)xy = 0$ is connected with these, and is not therefore an additional equation.

In this contact transformation the two spaces are symmetrically related; thus a point in either corresponds to a line of the tetrahedral complex in the other.

§ 212. We must now find what corresponds in space x, y, z to the plane

$$lx' + my' + nz' + k = 0.$$

Forming the equations of the Pfaffian system of which this plane is the generating surface we have

$$l + np' = 0, \quad m + nq' = 0,$$

and substituting for p' and q' in the equation

$$p'(b-c)zy + q'(c-a)zx - (a-b)xy = 0$$

of the contact transformation we have

$$(1) \quad l(b-c)yz+m(c-a)zx+n(a-b)xy = 0.$$

This, however, is not the only generating equation defining the M_2 which will correspond to the plane in the other space. For, eliminating y', z' from

$$\begin{aligned} axx'+byy'+czz'+d &= 0, \\ xx'+\ yy'+\ zz'+1 &= 0, \\ lx'+my'+\ nz'+k &= 0, \end{aligned}$$

we see that by aid of (1) x' disappears at the same time, and therefore all the three-rowed determinants vanish in the matrix

$$(2) \qquad \left\| \begin{matrix} ax, & by, & cz, & d \\ x, & y, & z, & 1 \\ l, & m, & n, & k \end{matrix} \right\|.$$

These are the equations of a twisted cubic, viz. the locus of a point whose polar planes with respect to the quadrics

$$x^2+y^2+z^2+1 = 0 \quad \text{and} \quad ax^2+by^2+cz^2+d = 0$$

intersect on the plane

$$lx+my+nz+k = 0.$$

This cubic passes through the origin and the points at infinity on the axes of coordinates.

To a plane in one space there will then correspond in the other space the twisted cubic given by the above equations. As a, b, c, d are fixed, when the contact transformation is fixed, we may call $l:m:n:k$ the coordinates of this twisted cubic.

§ 213. The coordinates of any point on this cubic are

$$x = \frac{l(t+d)}{k(t+a)}, \quad y = \frac{m(t+d)}{k(t+b)}, \quad z = \frac{n(t+d)}{k(t+c)}.$$

Since therefore the six coordinates of the line in space x', y', z' which corresponds to x, y, z are

$$\begin{aligned} l' &= (b-c)yz, \quad & m' &= (c-a)zx, \quad & n' &= (a-b)xy, \\ \alpha' &= (a-d)x, \quad & \beta' &= (b-d)y, \quad & \gamma' &= (c-d)z, \end{aligned}$$

the coordinates of the line which corresponds to a point on the twisted cubic are

$$l' = (b-c)mn(t+a)(t+d), \quad \alpha' = (a-d)lk(t+b)(t+c),$$

with similar expressions for the other coordinates.

The coordinates of the line joining two points on this twisted cubic are

$$l' = \frac{l\,(a-d)\,(t_1-t_2)}{k\,(t_1+a)\,(t_2+a)}, \qquad a' = \frac{mn\,(b-c)\,(t_1-t_2)\,(t_1+d)\,(t_2+d)}{k^2\,(t_1+b)\,(t_2+b)\,(t_1+c)\,(t_2+c)},$$

with similar expressions for m', n', β', γ'; such a line therefore belongs to the tetrahedral complex

$$\frac{l'a'}{(b-c)\,(a-d)} = \frac{m'\beta'}{(c-a)\,(b-d)} = \frac{n'\gamma'}{(a-b)\,(c-d)},$$

and so is divided in a constant ratio by the coordinate planes, and has, as its correspondent in space x', y', z', a point on the plane

$$lx' + my' + nz' + k = 0.$$

The twisted cubic which in one space corresponds to any plane in the other always passes through four fixed points, viz. the origin and the points at infinity on the axes of coordinates; and any straight line which intersects the cubic in two points is divided in a constant ratio by the coordinate planes. This ratio does not depend on the position of the plane which corresponds to the cubic.

It is generally true that *any* straight line intersecting *any* twisted cubic in two points is divided in a constant anharmonic ratio by the faces of any tetrahedron inscribed in the cubic. In order that a twisted cubic may belong to the family we are here considering it is only necessary that it should pass through the origin and the points at infinity on the axes and be such that the anharmonic ratio for this tetrahedron has the assigned value which defines the tetrahedral complex. We shall speak of these cubics as cubics of the given complex.

Since a plane can be drawn to pass through any three points we see that a twisted cubic can be drawn to intersect any three lines of the tetrahedral complex; for a line of this complex corresponds to a point in the other space.

§ 214. We next find what corresponds to the line

(1) $\quad a = mz' - ny', \quad \beta = nx' - lz', \quad \gamma = ly' - mx'.$

Eliminating y' and z' from the equations of this line and the given generating equations of the contact transformation, viz.

$$axx' + byy' + czz' + d = 0, \quad xx' + yy' + zz' + 1 = 0,$$

we get

$$(2)\quad \begin{aligned} x'(lx+my+nz)+l+\gamma y-\beta z &= 0,\\ x'(alx+bmy+cnz)+dl+b\gamma y-c\beta z &= 0. \end{aligned}$$

These are the equations of a generator of one system on the quadric

$$(3)\quad a(b-c)yz+\beta(c-a)zx+\gamma(a-b)xy \\ +l(a-d)x+m(b-d)y+n(c-d)z=0;$$

and since (2) corresponds to x', y', z' we see that this system (the first system, we shall call it) of generators on this quadric belongs to the tetrahedral complex.

Now any quadric passing through the origin and the points at infinity on the axes of coordinates is of the form (3); we thus have the following interesting theorem in geometry: the generators of a quadric are divided in a constant anharmonic ratio by the four planes of any inscribed tetrahedron *.

The following is an analytical proof not depending on contact transformation theory. The equation of the quadric referred to the tetrahedron as tetrahedron of reference is

$$a_1 yz+b_1 zx+c_1 xy+axw+byw+czw=0.$$

The conditions that the line

$$ly-mx=\gamma w, \quad nx-lz=\beta w$$

may lie wholly on the quadric are

$$a_1 mn+b_1 nl+c_1 lm=0, \quad a_1\beta\gamma-bl\gamma+cl\beta=0,$$
$$a_1(n\gamma-m\beta)+l(c_1\gamma-b_1\beta)+l(la+mb+nc)=0.$$

Eliminating l from these equations we get

$$(c_1 m^2\beta-b_1 n^2\gamma)(c_1 m+b_1 n) \\ +mn(c_1 bm^2+b_1 cn^2+(cc_1+bb_1-aa_1)mn)=0,$$
$$mn(c\beta-b\gamma)=\beta\gamma(c_1 m+b_1 n).$$

These equations give us to determine the ratio of β to γ

$$b_1 bn^2\gamma^2+c_1 cm^2\beta^2+(b_1 b+c_1 c-a_1 a)mn\beta\gamma=0;$$

and we have similar equations for $a:\beta$ and $a:\gamma$.

If the straight line intersects the faces of the tetrahedron

* This and much more about the tetrahedral complex will be found in *Berührungstransformationen*, Lie-Scheffers, Chap. VIII.

of reference in A, B, C, D respectively, and if the anharmonic ratio $\dfrac{AC\,.\,BD}{AD\,.\,BC}$ is denoted by λ, we therefore have

$$a_1 a\lambda^2-(a_1 a+b_1 b-c_1 c)\,\lambda+b_1 b=0,$$

so that the generator is divided in a constant ratio by the faces of the tetrahedron of reference.

§ 215. There are two systems of generators in the quadric

$$(1)\quad \alpha\,(b-c)\,yz+\beta\,(c-a)\,zx+\gamma\,(a-b)\,xy \\ +l\,(a-d)\,x+m\,(b-d)\,y+n\,(c-d)\,z=0.$$

To the first system of these generators we have seen that there correspond, in space x', y', z', the points on the lines

$$(2)\quad \alpha=mz'-ny',\quad \beta=nx'-lz',\quad \gamma=ly'-mx'.$$

The equations of the generators of the other system are

$$t\,(lx+my+nz)+alx+bmy+cnz=0,$$
$$t\,(l-\beta z+\gamma y)+b\gamma y-c\beta z+ld.$$

The six coordinates of this generator are given by

$$\alpha'=l\,(a+t),\quad \beta'=m\,(b+t),\quad \gamma'=n\,(c+t),$$
$$l'=\frac{\alpha\,(b+t)\,(c+t)}{d+t},\quad m'=\frac{\beta\,(c+t)\,(a+t)}{d+t},\quad n'=\frac{\gamma\,(a+t)\,(b+t)}{d+t};$$

and therefore to any generator of this system there corresponds in space x', y', z' the quadric

$$(3)\quad \alpha'(b-c)\,y'z'+\beta'(c-a)\,z'x'+\gamma'(a-b)\,x'y' \\ +l'(a-d)\,x'+m'(b-d)\,y'+n'(c-d)\,z'=0.$$

Since all generators of the first system intersect each generator of the second, we can conclude that all points lying on (2) must also lie on (3); that is, (3) contains the line (2); this may easily be verified directly.

§ 216. If the straight line whose coordinates are

$$l,\ m,\ n,\ \alpha,\ \beta,\ \gamma$$

belongs to the tetrahedral complex, that is, if

$$\frac{l\alpha}{(b-c)\,(a-d)}=\frac{m\beta}{(c-a)\,(b-d)}=\frac{n\gamma}{(a-b)\,(c-d)},$$

the quadric of the form (1) of § 215 which corresponds to the line is a cone.

The ∞^2 of elements which consists of points on the above line, together with the infinity of planes which contains the line, is therefore transformed into the cone M_2.

We know, however, that the M_2 which corresponds to a line of the tetrahedral complex is a point M_2, so that this point M_2 must coincide with the cone M_2. There is of course nothing paradoxical in this; for the point must be the vertex of the cone, and any plane through the vertex will be a tangent plane to the cone.

The quadric which corresponds to a straight line has, like the twisted cubic which corresponded to the plane, the properties of passing through the origin and the points at infinity on the axes of coordinates; it has also the property that its generators of one system are divided in the assigned ratio which defines the tetrahedral complex. We shall call any quadric of this family a quadric of the given complex.

The contact transformation we have now considered has the property of transforming point M_2's into the M_2's of lines of the tetrahedral complex; or, as we may briefly express it, points into lines of the complex. It also transforms planes into twisted cubics of the complex; and straight lines generally into quadrics of the complex, though, if the line belongs to the complex, the quadrics degenerate into points.

§ 217. We may now apply this method of transformation to deduce new theorems from theorems already known.

Thus a straight line can be drawn through any two points in space; therefore a quadric of the complex can be drawn through any two lines of the complex.

Again any two planes intersect in a straight line; therefore a quadric of the complex can be drawn through any two twisted cubics of the complex.

A straight line in space which intersects three fixed lines intersects an infinity of other fixed lines; therefore a quadric of the complex which touches three fixed quadrics of the complex touches also an infinity of fixed quadrics of the complex.

One more illustration of the method will be afforded by taking any six points P_1, P_2, P_3, P_4, P_5, P_6 on a twisted cubic of the complex; to these six points will correspond six lines of the complex, and all of these lines will lie on the plane which corresponds to the cubic. These lines are divided in a constant anharmonic ratio by the coordinate planes and the plane at infinity; and therefore are divided in a constant ratio by the sides of a fixed triangle. They

therefore all touch a parabola; let AB correspond to P_1, BC to P_2 and so on; B will then correspond to $P_1 P_2$. If we now apply Brianchon's theorem to the hexagon $ABCDEF$ formed by the six lines, we see that AD, BE, and CF are concurrent. To AD will correspond the quadric of the complex which contains the lines $P_1 P_6$ and $P_3 P_4$; to BE the quadric with the generators $P_1 P_2$ and $P_4 P_5$; to CF the quadric with the generators $P_2 P_3$ and $P_5 P_6$; the theorem which we can now deduce from Brianchon's is that these three quadrics have a common generator.

§ 218. We have now examined the first two classes of contact transformations and there remains the case where there are three generating equations; but as we can now express x', y', z' in terms of x, y, z, and x, y, z in terms of x', y', z', this is a mere extended point transformation. We have had examples of this class of contact transformation in Chapter II, and shall return to the subject in Chapter XX on differential invariants, so that we need not now consider it further.

CHAPTER XVIII

INFINITESIMAL CONTACT TRANSFORMATIONS

§ 219. If $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ are the coordinates of an element in n-way space,

$$\begin{aligned} z' &= z + t\zeta(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \\ x'_i &= x_i + t\xi_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \qquad (i = 1, \ldots, n), \\ p'_i &= p_i + t\pi_i(x_1, \ldots, x_n, z, p_1 \ldots, p_n) \end{aligned}$$

is an infinitesimal transformation of the elements, if t is a constant so small that its square may be neglected.

The transformation is an infinitesimal contact transformation if the Pfaffian equation

$$dz - p_1\,dx_1 - \ldots - p_n\,dx_n = 0$$

is unaltered; that is, if we have

$$dz' - \sum^{i=n} p'_i\,dx'_i = (1 + \rho t)\,(dz - \sum^{i=n} p_i\,dx_i),$$

where ρ is some function of the coordinates of the element.

Now $dz' = dz + td\zeta$, $dx'_i = dx_i + td\xi_i$, $dp'_i = dp_i + td\pi_i$;

if then we take $$W = \sum^{i=n} p_i\,\xi_i - \zeta,$$

we have

$$\begin{aligned} dz' - \sum^{i=n} p'_i\,dx'_i &= dz - \sum^{i=n} p_i\,dx_i + t\,(d\zeta - \sum^{i=n} p_i\,d\xi_i - \sum^{i=n} \pi_i\,dx_i) \\ &= dz - \sum^{i=n} p_i\,dx_i - tdW + t\sum^{i=n} (\xi_i\,dp_i - \pi_i\,dx_i) \end{aligned}$$

(neglecting small quantities of the order t^2); and therefore

$$\sum^{i=n} (\xi_i\,dp_i - \pi_i\,dx_i) - dW = \rho\,(dz - \sum^{i=n} p_i\,dx_i),$$

so that $\xi_i = \dfrac{\partial W}{\partial p_i}, \quad \rho = -\dfrac{\partial W}{\partial z}, \quad \pi_i = -\dfrac{\partial W}{\partial x_i} - p_i \dfrac{\partial W}{\partial z},$

$$\zeta = \sum^{i=n} p_i \xi_i - W = \sum^{i=n} p_i \frac{\partial W}{\partial p_i} - W.$$

§ 220. Conversely if W is any function whatever of the coordinates of an element,

$$(1) \quad x_i' = x_i + t\frac{\partial W}{\partial p_i}, \quad p_i' = p_i - t\left(\frac{\partial W}{\partial x_i} + p_i\frac{\partial W}{\partial z}\right),$$

$$z' = z + t\left(\sum^{i=n} p_i \frac{\partial W}{\partial p_i} - W\right)$$

will be an infinitesimal contact transformation; for

$$\begin{aligned} dz' - \sum^{i=n} p_i' dx_i' &= dz - t\,dW + t\sum^{i=n} p_i d\frac{\partial W}{\partial p_i} + t\sum^{i=n} \frac{\partial W}{\partial p_i} dp_i \\ &\qquad - \sum^{i=n} p_i dx_i + t\sum^{i=n} dx_i\left(\frac{\partial W}{\partial x_i} + p_i\frac{\partial W}{\partial z}\right) - t\sum^{i=n} p_i d\frac{\partial W}{\partial p_i} \\ &= dz - \sum^{i=n} p_i dx_i \\ &\qquad + t\left(-dW + \sum^{i=n}\frac{\partial W}{\partial p_i} dp_i + \sum^{i=n}\frac{\partial W}{\partial x_i}\partial x_i + \frac{\partial W}{\partial z}\sum^{i=n} p_i dx_i\right) \\ &= \left(1 - t\frac{\partial W}{\partial z}\right)\left(dz - \sum^{i=n} p_i dx_i\right). \end{aligned}$$

The function W is called the *characteristic* function of the infinitesimal contact transformation; and the corresponding infinitesimal operator is

$$\sum^{i=n}\frac{\partial W}{\partial p_i}\frac{\partial}{\partial x_i} - \sum^{i=n}\left(\frac{\partial W}{\partial x_i} + p_i\frac{\partial W}{\partial z}\right)\frac{\partial}{\partial p_i} + \sum^{i=n} p_i \frac{\partial W}{\partial p_i}\frac{\partial}{\partial z} - W\frac{\partial}{\partial z}.$$

If W does not contain z, and is homogeneous of the first degree in $p_1, \ldots, p_n$, the infinitesimal contact transformation is a homogeneous one.

§ 221. Suppose now that $\phi(z, x_1, \ldots, x_n, p_1, \ldots, p_n)$ is any function of the coordinates of an element, then $z', x_1', \ldots, x_n'$,

$p_1', \ldots, p_n'$ being the contiguous element defined by (1) of § 220,

$$\phi(z', x_1', \ldots, x_n', p_1', \ldots, p_n') = \phi + t[W, \phi] - tW\frac{\partial \phi}{\partial z},$$

where

$$[W, \phi] = \sum^{i=n} \left(\frac{\partial W}{\partial p_i}\left(\frac{\partial \phi}{\partial x_i} + p_i\frac{\partial \phi}{\partial z}\right) - \frac{\partial \phi}{\partial p_i}\left(\frac{\partial W}{\partial x_i} + p_i\frac{\partial W}{\partial z}\right)\right).$$

The necessary and sufficient condition therefore that the function ϕ should admit the infinitesimal contact transformation with the characteristic function W is

$$[W, \phi] = W\frac{\partial \phi}{\partial z}.$$

Similarly we see that the equation $\phi = 0$ admits the contact transformation if the equation $[W, \phi] - W\frac{\partial \phi}{\partial z} = 0$ is connected with $\phi = 0$.

If the equation $\phi = 0$ admits the contact transformation, with the characteristic function W, the equations $W = 0$ and $\phi = 0$ will be equations in involution.

§ 222. If $$\phi_1 = 0, \ldots, \phi_m = 0,$$ are any m equations in involution (§ 153), then, $W = 0$ being any equation connected with the system, this system will admit the contact transformation, whose characteristic function is W.

If we are given any function $\phi(z, x_1, \ldots, x_n, p_1, \ldots, p_n)$ of the coordinates of an element, we can find $2n$ unconnected functions in involution with this function; let these functions be

$$\phi_1(z, x_1, \ldots, x_n, p_1, \ldots, p_n), \ldots, \phi_{2n}(z, x_1, \ldots, x_n, p_1, \ldots, p_n);$$

it will now be proved that the equations

$$(1) \quad \phi_i(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = \phi_i(z^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0), \quad (i = 1, \ldots, 2n),$$

define a simple infinity of united elements, that is, an M_1 containing the assigned element $z^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0$.

If $$x_1, \ldots, x_n, z, p_1, \ldots, p_n$$
and $$x_1 + dx_1, \ldots, x_n + dx_n, z + dz, p_1 + dp_1, \ldots, p_n + dp_n$$

are two consecutive elements satisfying the equations (1) then

$$\frac{\partial \phi_i}{\partial x_1}dx_1+\ldots+\frac{\partial \phi_i}{\partial x_n}dx_n+\frac{\partial \phi_i}{\partial p_1}dp_1+\ldots+\frac{\partial \phi_i}{\partial p_n}\partial p_n+\frac{\partial \phi_i}{\partial z}dz=0; \tag{2}$$

and since all the functions $\phi_1, \ldots, \phi_{2n}$ are in involution with ϕ we must have

$$\begin{aligned} 0 = &\frac{\partial \phi_i}{\partial x_1}\frac{\partial \phi}{\partial p_1}+\ldots+\frac{\partial \phi_i}{\partial x_n}\frac{\partial \phi}{\partial p_n}+\frac{\partial \phi_i}{\partial z}\sum^{k=n} p_k\frac{\partial \phi}{\partial p_k} \\ &-\frac{\partial \phi_i}{\partial p_1}\left(\frac{\partial \phi}{\partial x_1}+p_1\frac{\partial \phi}{\partial z}\right)-\ldots-\frac{\partial \phi_i}{\partial p_n}\left(\frac{\partial \phi}{\partial x_n}+p_n\frac{\partial \phi}{\partial z}\right). \end{aligned} \tag{3}$$

There are $2n$ equations of the form (2) by means of which we can determine the ratios of

$$dx_1, \ldots, dx_n, \; dz, \; dp_1, \ldots, dp_n:$$

the equations (3) to determine the ratios of

$$\frac{\partial \phi}{\partial p_1}, \ldots, \frac{\partial \phi}{\partial p_n}, \quad \sum^{k=n} p_k\frac{\partial \phi}{\partial p_k}, \quad -\frac{\partial \phi}{\partial x_1}-p_1\frac{\partial \phi}{\partial z}, \ldots, -\frac{\partial \phi}{\partial x_n}-p_n\frac{\partial \phi}{\partial z},$$

are exactly the same; and therefore we conclude that

$$\begin{aligned} &\frac{dx_1}{\frac{\partial \phi}{\partial p_1}}=\ldots=\frac{dx_n}{\frac{\partial \phi}{\partial p_n}}=\frac{dz}{\sum^{k=n} p_k\frac{\partial \phi}{\partial p_k}} \\ &\qquad =\frac{dp_1}{-\frac{\partial \phi}{\partial x_1}-p_1\frac{\partial \phi}{\partial z}}=\ldots=\frac{dp_n}{-\frac{\partial \phi}{\partial x_n}-p_n\frac{\partial \phi}{\partial z}}. \end{aligned} \tag{4}$$

Since the equations (4) satisfy Pfaff's equation

$$dz = p_1dx_1+\ldots+p_ndx_n,$$

we conclude that the infinity of elements satisfying each of the equations (1) consists of united elements.

Any simple infinity of elements satisfying the equations (4) is called a *characteristic* manifold or M_1 of the function ϕ.

It is possible to describe one, and only one, of these characteristic M_1's through any assigned element of space $z^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0$; and it is easily seen to lie altogether on the manifold

$$\phi(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = \phi(z^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0),$$

as well as on each of the manifolds given by (1).

We shall now prove that by any contact transformation a characteristic M_1 of a function is transformed into a characteristic M_1 of the corresponding function. This follows at once from the facts: (1) that two functions in involution are transformed into two functions in involution; and (2) that the characteristic M_1 of a function ϕ, which contains the element $z^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0$, consists of all elements common to

$$\phi_i(z, x_1, \ldots, x_n, p_1, \ldots, p_n) = \phi_i(z^0, x_1^0, \ldots, x_n^0, p_1^0, \ldots, p_n^0),$$
$$(i = 1, \ldots, 2n),$$

where $\phi_1, \ldots, \phi_{2n}$ are any $2n$ unconnected functions in involution with ϕ.

§ 223. We may now interpret an infinitesimal contact transformation as follows: take any element $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ and construct the characteristic M_1 of the characteristic function W which contains this element. Imagine an element to be moving along this M_1, the consecutive element to the one assumed is

$$z + t\dot{z},\ x_1 + t\dot{x}_1, \ldots, x_n + t\dot{x}_n,\ p_1 + t\dot{p}_1, \ldots, p_n + t\dot{p}_n,$$

where t is the small interval of time taken to move to this consecutive position; the infinitesimal contact transformation which corresponds to W is then given by

$$z' = z + t\zeta,\ x_1' = x_1 + t\xi_1, \ldots, x_n' = x_n + t\xi_n,$$
$$p_1' = p_1 + t\pi_1, \ldots, p_n' = p_n + t\pi_n,$$

where

$$\dot{x}_1 = \xi_1, \ldots, \dot{x}_n = \xi_n,\ \dot{p}_1 = \pi_1, \ldots, \dot{p}_n = \pi_n, \text{ but } \dot{z} - W = \zeta.$$

We may then say that the velocity of an element, under the effect of the infinitesimal contact transformation whose characteristic is W, is composed of a velocity along the characteristic M_1 of W containing this element, and a velocity along the axis of z; the ratio of the z component of the first velocity to that of the second being as

$$\sum^{k=n} p_k \frac{\partial W}{\partial p_k} \text{ to } -W.$$

§ 224. If P and P' are two consecutive points in space, the straight line joining the points and terminated by them is called a *linear element*.

If we take any point $z, x_1, \ldots, x_n$ then ∞^{n-1} elements $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ pass through this point, and satisfy the equation $\phi = 0$; it therefore follows that ∞^{n-1} characteristic M_1's of this equation pass through any point. Taking

$$z, x_1, \ldots, x_n, \; dz : dx_1 : dx_2 : \ldots : dx_n$$

to be the coordinates of the linear element joining $z, x_1, \ldots, x_n$ to a consecutive point on any one of these characteristic M_1's, we see that these coordinates must satisfy the equation (or equations) obtained by eliminating $p_1, \ldots, p_n$ from the equations

$$(1) \quad \frac{dx_1}{\frac{\partial \phi}{\partial p_1}} = \frac{dx_2}{\frac{\partial \phi}{\partial p_2}} = \ldots = \frac{dx_n}{\frac{\partial \phi}{\partial p_n}} = \frac{dz}{p_1 \frac{\partial \phi}{\partial p_1} + \ldots + p_n \frac{\partial \phi}{\partial p_n}}, \quad \phi = 0.$$

This equation is called the equation of *the elementary integral cone of* $\phi = 0$ at the point $x_1, \ldots, x_n, z$.

We have seen that if the equation $\phi = 0$ is transformed by a contact transformation into $\psi = 0$, then the characteristic M_1's of $\phi = 0$ are transformed so as to be the characteristic M_1's of $\psi = 0$. It does not, however, follow that the elementary integral cones of $\phi = 0$ will be transformed into the elementary integral cones of $\psi = 0$; for characteristic M_1's, meeting in a point, will not in general be transformed to characteristic M_1's, meeting in a point.

If, however, the transformation is merely a point transformation, the elementary integral cones of one equation will be transformed to the elementary integral cones of the other. In particular, the point transformations which leave a given equation of the first order unaltered, will also leave the system of integral cones unaltered, though naturally these cones will be transformed *inter se.*

Looking on

$$p_1 dx_1 + \ldots + p_n dx_n = dz$$

as the equation of an elementary plane whose coordinates are $p_1, \ldots, p_n$, we easily prove that $\phi = 0$ is the tangential equation of the elementary integral cone of $\phi = 0$ at the point $z, x_1, \ldots, x_n$.

Conversely, suppose we are given an equation, homogeneous in $dz, dx_1, \ldots, dx_n$, and connecting $z, x_1, \ldots, x_n, dz, dx_1, \ldots, dx_n$, the coordinates of a linear element; then, if, regarding $dz : dx_1 : dx_2 : \ldots$ as the variables, we find its tangential equation, we shall have a differential equation of the first order,

of which the given equation will be an elementary integral cone.

We thus see that any point transformation, which leaves a differential equation of the first order unaltered, will also leave unaltered an equation between the coordinates of a linear element; and, conversely, a point transformation, which leaves an equation between the coordinates of a linear element unaltered, will also leave unaltered a differential equation of the first order.

An equation between the coordinates of a linear element is called a *Mongian equation.* We have now proved that to every Mongian equation there will correspond in general one differential equation of the first order; and conversely to every differential equation of the first order there will in general correspond a Mongian equation.

We say, 'in general,' because, for instance, if the elementary integral cone at a point shrinks into a line (as it would if the given differential equation were linear) there would not be one definite Mongian equation but the several equations which make up the line; and other cases might arise where the result of eliminating $p_1, \ldots, p_n$ from (1) would be several equations.

So also if the Mongian equation were linear in $dz, dx_1, \ldots, dx_n$ instead of having one equation between the coordinates $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, we should have n such equations; for the envelope of a plane touching a given plane is the plane itself.

§ 225. *Example.* We saw in § 33 that the point transformations which were admitted by

$$1+p^2+q^2 = 0,$$

were also admitted by

$$dx^2+dy^2+dz^2 = 0,$$

the equation satisfied by the linear element of a minimum curve; these two equations are clearly associated in the manner just described.

A straight line of the tetrahedral complex which we considered in Chapter XVII has its linear elements connected by the equation,

$$(1)\quad \begin{aligned}(b-c)(a-d)x\,dy\,dz+(c-a)(b-d)y\,dz\,dx \\ +(a-b)(c-d)z\,dx\,dy = 0.\end{aligned}$$

If we form the associated partial differential equation, by expressing the condition that

$$pdx+qdy=dz$$

may, when we substitute $pdx+qdy$ for dz in (1), lead to a quadratic with equal roots in $dx:dy$, we obtain

$$(2) \quad \begin{aligned}(px(a-d)(b-c)+qy(b-d)(c-a)+(c-d)(a-b))^2 \\ = 4pqxy(a-d)(b-c)(b-d)(c-a),\end{aligned}$$

which may also be written in the form

$$\sqrt{px(a-d)(b-c)}+\sqrt{qy(b-d)(c-a)}+\sqrt{(c-d)(b-a)}=0.$$

We could now find the group—assuming such to exist—of point transformations admitted by (1), and the group admitted by (2); and seeing that these coincide we should verify the general theorem of their coincidence.

Without, however, actually finding either of these groups, we may easily verify that the point transformation

$$(3) \quad \begin{aligned} x &= e^{\frac{y'}{\sqrt{(c-a)(b-d)}}+\frac{z'}{\sqrt{(a-b)(c-d)}}}, \\ y &= e^{\frac{z'}{\sqrt{(a-b)(c-d)}}+\frac{x'}{\sqrt{(b-c)(a-d)}}}, \\ z &= e^{\frac{x'}{\sqrt{(b-c)(a-d)}}+\frac{y'}{\sqrt{(c-a)(b-d)}}}, \end{aligned}$$

transforms

$$(b-c)(a-d)xdydz+(c-a)(b-d)ydzdx \\ +(a-b)(c-d)zdxdy=0$$

into $$dx'^2+dy'^2+dz'^2=0.$$

The group found in Chapter II will therefore, when the transformation (3) is applied to it, be a group transforming any linear element of a tetrahedral complex into another such linear element; and will therefore leave unaltered the equation (1). It may also be easily verified that (3) will transform (2) into

$$1+p'^2+q'^2=0.$$

We can always find a contact transformation which will transform any given partial differential equation into any other assigned equation, if both are of the first order; this we have proved in § 183; but it is not generally true that we can find a point transformation which will do so. The

example which we have just considered, suggests that if we wish to determine whether two assigned equations can be transformed, the one into the other, by a point transformation, it may be more convenient to determine whether or no the corresponding Mongian equations are transformable into one another by a point transformation.

§ 226. Let $\overline{W}$ denote the infinitesimal operator which corresponds to the characteristic function W, viz.

$$\sum^{i=n}\frac{\partial W}{\partial p_i}\frac{\partial}{\partial x_i}-\sum^{i=n}\left(\frac{\partial W}{\partial x_i}+p_i\frac{\partial W}{\partial z}\right)\frac{\partial}{\partial p_i}+\sum^{i=n}p_i\frac{\partial W}{\partial p_i}\frac{\partial}{\partial z}-W\frac{\partial}{\partial z}.$$

As we vary the characteristic function we get different operators; we must now find the alternant of two such operators.

To do this, we take

$$y_1=x_1,\ldots,y_n=x_n,\quad y_{n+1}=z,$$

$$p_1=-\frac{q_1}{q_{n+1}},\ldots,p_n=-\frac{q_n}{q_{n+1}},\quad H=-q_{n+1}W,$$

and we find the operator in the variables

$$y_1,\ldots,y_{n+1},\ q_1,\ldots,q_{n+1},$$

which has the same effect on any function of these variables (provided that it is homogeneous and of zero degree) as the operator $\overline{W}$ has on the same function expressed in terms of $x_1,\ldots,x_n,\ z,\ p_1,\ldots,p_n$.

Let the function on which we are to operate be

$$\phi(x_1,\ldots,x_n,\ z,\ p_1,\ldots,p_n)\equiv\psi(y_1,\ldots,y_{n+1},\ q_1,\ldots,q_{n+1}),$$

then by § 184

$$\frac{\partial\phi}{\partial p_i}=-q_{n+1}\frac{\partial\psi}{\partial q_1},\quad\frac{\partial\phi}{\partial x_i}=\frac{\partial\psi}{\partial y_i},\quad\frac{\partial\phi}{\partial z}=\frac{\partial\psi}{\partial y_{n+1}};$$

$$(i=1,\ldots,n),$$

and, since ψ is homogeneous of zero degree,

$$\sum^{i=n}p_i\frac{\partial\phi}{\partial p_i}=\sum^{i=n}q_i\frac{\partial\psi}{\partial q_i}=-q_{n+1}\frac{\partial\psi}{\partial q_{n+1}}.$$

We now get

$$\overline{W}\phi = -q_{n+1}\sum^{i=n}\frac{\partial}{\partial q_i}\left(\frac{-H}{q_{n+1}}\right)\frac{\partial\psi}{\partial y_i} + q_{n+1}\sum^{i=n}\frac{\partial}{\partial y_i}\left(\frac{-H}{q_{n+1}}\right)\frac{\partial\psi}{\partial q_i}$$

$$+q_{n+1}\frac{\partial}{\partial y_{n+1}}\left(\frac{-H}{q_{n+1}}\right)\frac{\partial\psi}{\partial q_{n+1}} - q_{n+1}\frac{\partial}{\partial q_{n+1}}\left(\frac{-H}{q_{n+1}}\right)\frac{\partial\psi}{\partial y_{n+1}}$$

$$+\frac{H}{q_{n+1}}\frac{\partial\psi}{\partial y_{n+1}};$$

and therefore

$$\overline{W}\phi = \sum^{i=n+1}\frac{\partial H}{\partial q_i}\frac{\partial\psi}{\partial y_i} - \sum^{i=n+1}\frac{\partial H}{\partial y_i}\frac{\partial\psi}{\partial q_i} = \overline{H}\psi,$$

where $\overline{H}$ is the infinitesimal homogeneous contact operator which corresponds to the characteristic function H.

That is, $\overline{W}$ operating on any function of $x_1, \ldots, x_n\ z, p_1, \ldots, p_n$ has the same effect as $\overline{H}$ on the equivalent function of $y_1, \ldots, y_{n+1}, q_1, \ldots, q_{n+1}$ where $H = -q_{n+1}W$.

It therefore follows that

$$\overline{W}_1\overline{W}_2 - \overline{W}_2\overline{W}_1 = \overline{H}_1\overline{H}_2 - \overline{H}_2\overline{H}_1 = \overline{(H_1, H_2)}.$$

We proved in § 184 that W_1 and W_2 being any functions of $x_1, \ldots, x_n, z, p_1, \ldots, p_n$

$$[W_1, W_2]_{z,x,p} = -q_{n+1}(W_1, W_2)_{y,q};$$

and therefore

$$\frac{1}{-q_{n+1}}(H_1, H_2)_{y,q} = \frac{-1}{q_{n+1}}(q_{n+1}W_1,\ q_{n+1}W_2)_{y,q}$$

$$= (-q_{n+1}W_1, W_2)_{y,q} + W_2\frac{\partial W_1}{\partial y_{n+1}}$$

$$= -q_{n+1}(W_1, W_2)_{y,q} - \left(W_1\frac{\partial W}{\partial y_{n+1}} - W_2\frac{\partial W_1}{\partial y_{n+1}}\right)$$

$$= [W_1, W_2]_{z,x,p} - \left(W_1\frac{\partial W_2}{\partial z} - W_2\frac{\partial W_1}{\partial z}\right).$$

That is, $\overline{W}_1\overline{W}_2 - \overline{W}_2\overline{W}_1$ has the characteristic function

$$[W_1, W_2]_{z,x,p} - \left(W_1\frac{\partial W_2}{\partial z} - W_2\frac{\partial W_1}{\partial z}\right).$$

§ 227. We next proceed to show how the operator $\overline{W}$ is transformed by the contact transformation

(1) $$x_i' = X_i, \quad z' = Z, \quad p_i' = P_i,$$

with the multiplier ρ defined by

$$dZ - \sum^{i=n} P_i \, dX_i = \rho \, (dz - \sum^{i=n} p_i \, dx_i).$$

Take

(2) $$\begin{aligned} x_1 &= y_1, \ldots, x_n = y_n, \ z = y_{n+1}, \ p_1 = \frac{-q_1}{q_{n+1}}, \ldots, p_n = \frac{-q_n}{q_{n+1}}, \\ x_1' &= y_1', \ldots, x_n' = y_n', \ z' = y_{n+1}', \ p_1' = \frac{-q_1'}{q_{n+1}'}, \ldots, \rho = \frac{q_{n+1}}{q_{n+1}'}; \end{aligned}$$

and let $\quad y_i' = Y_i, \quad q_i' = Q_i, \qquad (i = 1, \ldots, n+1)$

be the homogeneous contact transformation equivalent to (1) obtained by eliminating x, p and x', p' from (1) and (2).

Let $H = -q_{n+1} W$; let K' denote the function of y', q' equivalent to H; and let V' be that function of x', p' which is given by $K' = -q_{n+1}' V'$.

We now have $H = K'$ and therefore by § 183 $\overline{H} = \overline{K'}$; and having proved that $\overline{W} = \overline{H}$, and $\overline{V'} = \overline{K'}$, we conclude that $\overline{W} = \overline{V'}$.

Now $V' = \dfrac{q_{n+1}}{q_{n+1}'} W = \rho W$; in order therefore to express $\overline{W}$ in terms of the variables $x_1', \ldots, x_n', z', p_1', \ldots, p_n'$ we find ρ, and then express ρW in terms of these variables by (1); the function thus obtained will be the characteristic function, with respect to the new variables, of the required operator, equivalent to $\overline{W}$.

§ 228. The totality of contact transformations form a group. For, $z', x_1', \ldots, x_n', p_1', \ldots, p_n'$ being the element derived by any contact transformation from $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, and $z'', x_1'', \ldots, x_n'', p_1'', \ldots, p_n''$ being similarly derived from $z', x_1', \ldots, x_n', p_1', \ldots, p_n'$ by any other contact transformation, we deduce from

$$dz' - \sum^{i=n} p_i' \, dx_i' = \rho \, (dz - \sum^{i=n} p_i \, dx_i),$$

and $$dz'' - \sum^{i=n} p_i'' \, dx_i'' = \rho' \, (dz' - \sum^{i=n} p_i' \, dx_i'),$$

that $$dz'' - \sum_{i=n} p_1'' dx_i'' = \rho\rho' (dz - \sum_{i=n} p_i dx_i).$$

Therefore z'', $x_1'', \ldots, x_n''$, $p_1'', \ldots, p_n''$ is derived from

$$z,\ x_1, \ldots, x_n,\ p_1, \ldots, p_n$$

by a contact transformation; that is, contact transformations satisfy the definition of a group, and clearly, the group is a continuous one.

We are now going to explain what is meant by a *finite continuous contact group*; it will be seen that many of the properties of finite continuous point groups can be transferred to the groups now about to be defined.

If
$$\begin{aligned} x_i' &= X_i (x_1, \ldots, x_n,\ z,\ p_1, \ldots, p_n,\ a_1, \ldots, a_r), \\ p_i' &= P_i (x_1, \ldots, x_n,\ z,\ p_1, \ldots, p_n,\ a_1, \ldots, a_r), \\ z' &= Z (x_1, \ldots, x_n,\ z,\ p_1, \ldots, p_n,\ a_1, \ldots, a_r) \end{aligned}$$

is a contact transformation for all values of the constants $a_1, \ldots, a_r$; and if from these equations and

$$\begin{aligned} x_i'' &= X_i (x_1', \ldots, x_n',\ z',\ p_1', \ldots, p_n',\ b_1, \ldots, b_r), \\ p_i'' &= P_i (x_1', \ldots, x_n',\ z',\ p_1', \ldots, p_n',\ b_1, \ldots, b_r), \\ z'' &= Z (x_1', \ldots, x_n',\ z',\ p_1', \ldots, p_n',\ b_1, \ldots, b_r), \end{aligned}$$

where $b_1, \ldots, b_r$ are another set of constants, we can deduce

$$\begin{aligned} x_i'' &= X_i (x_1, \ldots, x_n,\ z,\ p_1, \ldots, p_n,\ c_1, \ldots, c_r), \\ p_i'' &= P_i (x_1, \ldots, x_n,\ z,\ p_1, \ldots, p_n,\ c_1, \ldots, c_r), \\ z'' &= Z (x_1, \ldots, x_n,\ z,\ p_1, \ldots, p_n,\ c_1, \ldots, c_r), \end{aligned}$$

where $c_1, \ldots, c_r$ are constants depending on $a_1, \ldots, a_r$, $b_1, \ldots, b_r$, then X_i, P_i, Z are said to be functions defining a finite continuous contact transformation group.

Such a group will have r independent infinitesimal operators $\overline{W}_1, \ldots, \overline{W}_r$. We see at once that the corresponding characteristic functions must be independent, that is, there must be no relation of the form

$$c_1 W_1 + \ldots + c_r W_r = 0,$$

where $c_1, \ldots, c_r$ are constants, connecting the characteristic functions. Also any finite transformations of the group can be obtained by endless repetition of the proper infinitesimal transformation.

The alternant of any two of these operators is not independent of the set of operators; we must therefore have

$$(\overline{W}_i, \overline{W}_k) \equiv \sum^{h=r} c_{ikh} \overline{W}_h, \qquad \left(\begin{matrix} i = 1, \ldots, r \\ k = 1, \ldots, r \end{matrix}\right).$$

Conversely, if we have r independent operators satisfying these conditions, they generate a finite continuous contact transformation group. If we use the symbol $\{W_i, W_k\}$ to denote $[W_i, W_k]_{z,x,p} - W_1 \dfrac{\partial W_2}{\partial z} + W_2 \dfrac{\partial W_1}{\partial z}$, we can express this fundamental theorem in terms of the characteristic functions thus:

$$\{W_i, W_k\} \equiv \sum^{h=r} c_{ikh} W_h.$$

These theorems for contact groups follow at once from what has been proved for point groups.

The constants $c_{ikh}, \ldots$ are still called the structure constants of the group.

§ 229. If W is of the particular form

$$p_1 \xi_1 + \ldots + p_n \xi_n - \zeta,$$

where $\xi_1, \ldots, \xi_n, \zeta$ involve only $x_1, \ldots, x_n, z$, the corresponding operator is said to be the *extended operator* of

$$\xi_1 \frac{\partial}{\partial x_1} + \ldots + \xi_n \frac{\partial}{\partial x_n} - \zeta \frac{\partial}{\partial z};$$

and

$$\begin{aligned} Z' &= z + t\zeta(x_1, \ldots, x_n, z) \\ x_i' &= x_i + t\xi_i(x_1, \ldots, x_n, z) \\ p_i' &= p_i + t\pi_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n) \end{aligned}$$

is said to be the *extended infinitesimal point transformation* of $x_1' = x_1 + t\xi_1(x_1, \ldots, x_n, z), \ldots, x_n' = x_n + t\xi_n(x_1, \ldots, x_n, z),$

$$z' = z + t\zeta(x_1, \ldots, x_n, z),$$

and it is entirely given when the point transformation is given.

Suppose that

$$(1) \quad x_1' = X_1(x_1, \ldots, x_n, z, a_1, \ldots, a_r), \ldots,$$
$$x_n' = X_n(x_1, \ldots, x_n, z, a_1, \ldots, a_r), \; z' = Z(x_1, \ldots, x_n, z, a_1, \ldots, a_r)$$

are the equations of a point group; when we know the form

of the functions $X_1, \ldots, X_n, Z$ we can, as in § 185, find the form of the functions $P_1, \ldots, P_n$ where

$$p_i' = P_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n, a_1, \ldots, a_r), \qquad (i = 1, \ldots, n).$$

It is now obvious that in the variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ these $(2n+1)$ equations define a group of order r; for, from (1) and

$$(2) \qquad x_i'' = X_i(x_1', \ldots, x_n', z', b_1, \ldots, b_r),$$
$$z'' = Z(x_1', \ldots, x_n', z, b_1, \ldots, b_r), \qquad (i = 1, \ldots, n),$$

where $b_1, \ldots, b_r$ are constants, and where the equations (2) involve the additional equations

$$p_i'' = P_i(x_1', \ldots, x_n', z', p_1', \ldots, p_n', b_1, \ldots, b_n), \qquad (i = 1, \ldots, n),$$

we may deduce

$$(3) \qquad x_i'' = X_i(x_1, \ldots, x_n, z, c_1, \ldots, c_r),$$
$$z' = Z(x_1, \ldots, x_n, z, c_1, \ldots, c_r), \qquad (i = 1, \ldots, n),$$

where $c_1, \ldots, c_r$ are constants which are functions of the sets $a_1, \ldots, a_r$, $b_1, \ldots, b_r$; and from (3) we may deduce

$$p_i'' = P_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n, c_1, \ldots, c_r), \qquad (i = 1, \ldots, n).$$

§ 230. Let $\overline{W}_1, \ldots, \overline{W}_r$ be the extended operators of this group in the $2n+1$ variables, and $\overline{U}_1, \ldots, \overline{U}_r$ the operators of the original group; it can now be proved that the structure constants of the extended group are the same as the structure constants of the original one.

Let
$$(\overline{W}_i, \overline{W}_k) = \sum^{h=r} \gamma_{ikh} \overline{W}_h,$$
$$(\overline{U}_i, \overline{U}_k) = \sum^{h=r} c_{ikh} \overline{U}_h,$$

and let
$$\overline{W}_i = \overline{U}_i + \overline{V}_i,$$

so that in $\overline{V}_i$ the terms $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}, \dfrac{\partial}{\partial z}$ do not occur.

We now have

$$(\overline{W}_i, \overline{W}_k) = (\overline{U}_i + \overline{V}_i, \overline{U}_k + \overline{V}_k)$$
$$= (\overline{U}_i, \overline{U}_k) + \text{operators in } \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \text{ only,}$$

for the coefficients of $\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}, \dfrac{\partial}{\partial z}$ in $\overline{U}_i$ and $\overline{U}_k$ involve only $x_1, \ldots, x_n, z$.

We have, therefore,

$$\sum^{h=r} \gamma_{ikh} \overline{W}_h = \sum^{h=r} c_{ikh} \overline{U}_h + \text{operators in } \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_r} \text{ only};$$

so that

$$\sum^{h=r} (\gamma_{ikh} - c_{ikh}) \overline{U}_h = \text{operators not involving } \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial z}.$$

It follows that $\gamma_{ikh} \equiv c_{ikh}$ for all values of i, k, h; that is, the extended group has the same structure constants as the original point group.

We see, therefore, that if we are given any structure constants, we can always find at least one contact group (viz. the extended point group) with the assigned structure; and, therefore, the third fundamental theorem also holds for contact transformation groups.

§ 231. We now proceed to obtain the necessary and sufficient conditions that two groups of contact transformations, in the same number of variables, may be transformable, the one into the other, by a contact transformation. Since a contact transformation in $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ can be expressed as a homogeneous transformation in $y_1, \ldots, y_{n+1}, q_1, \ldots, q_{n+1}$, it will be sufficient to consider this problem for the case of the homogeneous contact groups.

Suppose $H_1, \ldots, H_r$ are the r independent characteristic functions of a finite continuous homogeneous group; let us apply any homogeneous transformation, and let these functions become respectively $K'_1, \ldots, K'_r$ when expressed in terms of the new variables $y'_1, \ldots, y'_n, q'_1, \ldots, q'_n$ by the given homogeneous contact transformation

$$y'_i = Y_i(y_1, \ldots, y_n, q_1, \ldots, q_n), \quad q'_i = Q_i(y_1, \ldots, y_n, q_1, \ldots, q_n),$$
$$(i = 1, \ldots, n).$$

We know that $(H_i, H_j)_{y,q} = (K'_i, K'_j)_{q',y'}$, and therefore

$$(K'_i, K'_j)_{y',q'} = \sum^{h=r} c_{ijh} K'_h;$$

so that the new characteristic functions in $y'_1, \ldots, y'_n, q'_1, \ldots, q'_n$, generate a group with the same structure constants.

Now the functions $H_1, \ldots, H_r$ are independent in the sense that there is no relation between them of the form

$$c_1 H_1 + \ldots + c_r H_r \equiv 0,$$

where $c_1, \ldots, c_r$ are constants; but they do not need to be functionally unconnected. Suppose that $H_1, \ldots, H_m$ are functionally unconnected, and that the other functions $H_{m+1}, \ldots, H_r$ can be expressed in terms of them, so that

$$H_{m+t} \equiv \phi_{m+t}(H_1, \ldots, H_m), \qquad (t = 1, \ldots, r-m),$$

and therefore

$$K'_{m+t} \equiv \phi_{m+t}(K'_1, \ldots, K'_m).$$

If then we are given the r characteristic functions of a transformation group, viz. $H_1, \ldots, H_r$, and the r characteristic functions of another group, viz. $K_1, \ldots, K_r$, we cannot transform the one group into the other, so that H_i may become K_i, unless the structure constants are the same, and unless the functional relations are also the same.

§ 232. We shall now prove that these necessary conditions are sufficient. Let $H_1, \ldots, H_r$ be the one independent set of characteristic functions such that

$$(H_i, H_j) = \sum^{h=r} c_{ijh} H_h,$$

and $\quad H_{m+t} \equiv \phi_{m+t}(H_1, \ldots, H_m), \qquad (t = 1, \ldots, r-m)$;

and let $K_1, \ldots, K_r$ be another set of independent characteristic functions such that

$$(K_i, K_j) = \sum^{h=r} c_{ijh} K_h,$$

and $\quad K_{m+t} = \phi_{m+t}(K_1, \ldots, K_m), \qquad (t = 1, \ldots, r-m).$

$H_1, \ldots, H_m$ now form a homogeneous function system with the structure functions $w_{ij}, \ldots, w_i, \ldots$ where

$$w_{ij} = \sum^{s=m} c_{ijs} H_s + \sum^{t=r-m} c_{i,j,m+t}\, \phi_{m+t}(H_1, \ldots, H_m), \quad w_i = 1,$$
$$\binom{i = 1, \ldots, m}{j = 1, \ldots, m}.$$

By what we have proved in § 182 there can now be found a homogeneous contact transformation, which will transform $H_1, \ldots, H_m$ into $K_1, \ldots, K_m$ respectively, since the two systems have the same structure functions.

It is clear that this transformation will also transform $H_{m+1}, \ldots, H_r$ into $K_{m+1}, \ldots, K_r$ respectively; the necessary conditions are therefore also sufficient conditions.

It might be supposed that we could from this theorem

deduce the condition that two point groups should be transformable, the one into the other; viz. that all we should have to do would be to extend the point groups, and then see whether they were so transformable. We could not infer from this, however, that the point groups would be transformable into one another by a point transformation, unless we know that the contact transformation, which transforms the one extended point group into the other extended point group, is itself a mere extended point transformation.

§ 233. We have proved that given any system of structure constants we can always find a contact group with the given structure. The particular one we have shown how to construct was an extended point group; there will however be others; in fact, we have only to apply an arbitrary contact transformation to this extended point group, and we shall have a group which will not generally be a mere extended point group. Such contact groups, however, being deducible from extended point groups by a contact transformation, are said to be *reducible contact transformation groups*; other groups which have not this property are said to be *irreducible.*

The structure constants of any contact transformation group, reducible or otherwise, satisfy the conditions

$$c_{ikj}+c_{kij}=0,$$

$$\sum^{t=n}(c_{ikt}\,c_{tjs}+c_{kjt}\,c_{tis}+c_{jit}\,c_{tks})=0,$$

as we at once see from the identities

$$(\overline{W}_i,\overline{W}_k)+(\overline{W}_k,\overline{W}_i)\equiv 0,$$

$$((\overline{W}_i,\overline{W}_k),\overline{W}_j)+((\overline{W}_k,\overline{W}_j),\overline{W}_i)+((\overline{W}_j,\overline{W}_i),\overline{W}_k)\equiv 0.$$

§ 234. Contact transformation groups in $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ are point groups in these $(2n+1)$ variables; but it is not true, conversely, that point groups in $(2n+1)$ variables are necessarily, or generally, contact transformation groups. If we write the variables in the form $z, x_1, \ldots, x_n, p_1, \ldots, p_n$, the group in these variables will only be a contact transformation one in the $(n+1)$-way space $z, x_1, \ldots, x_n$ if all the transformations of the group are characterized by the property of leaving the equation

$$dz-p_1dx_1-\ldots-p_ndx_n=0$$

invariant.

From a knowledge then of contact transformation groups

in spaces of lower dimensions we can often deduce important information as to point groups in space of higher dimensions. Thus suppose, in space of s dimensions, we know that a group, which we wish to determine, has the property of leaving unaltered an equation of the form

$$f_1 dx_1 + \dots + f_s dx_s = 0,$$

where $f_1, \dots, f_s$ are functions of $x_1, \dots, x_s$. By the theory of Pfaff's Problem a transformation of the variables will reduce this equation to one or other of the two forms

$$dy_{m+1} - p_1 dy_1 - \dots - p_m dy_m = 0,$$
$$p_1 dy_1 + \dots + p_m dy_m = 0,$$

where $2m+1$ does not exceed s; and therefore the group we seek must, when expressed in terms of the new variables, be a contact transformation group in a space of not more than $\frac{1}{2}(s+1)$ dimensions.

CHAPTER XIX

THE EXTENDED INFINITESIMAL CONTACT TRANSFORMATIONS: APPLICATIONS TO GEOMETRY

§ 235. If $z = \phi(x_1, \ldots, x_n)$ is any surface in $(n+1)$-way space, we shall now consider how the derivatives of z with respect to $x_1, \ldots, x_n$ are transformed by the application of an assigned infinitesimal contact transformation.

We must regard the function ϕ which defines the surface as unknown; for otherwise the derivatives of z would be known functions of $x_1, \ldots, x_n$; and the contact transformation would be (when we replace $p_1, \ldots, p_n$ by their expressions in terms of $x_1, \ldots, x_n$ obtained from $z = \phi(x_1, \ldots, x_n)$) a mere point transformation; and would apply, not to any surface, but merely to the particular surface under consideration.

Let $p_1, \ldots, p_n$ be the first derivatives, $p_{ij}, \ldots$ the second derivatives, where p_{ij} denotes $\dfrac{\partial^2 z}{\partial x_i \partial x_j}$, $p_{ijk}, \ldots$ the third derivatives and so on; and let W be the characteristic function of the assigned contact transformation which it is our object to extend to derivatives of any required order.

Let the extended contact transformation be denoted by

$$\begin{aligned}
z' &= z + t\zeta(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \\
x'_i &= x_i + t\xi_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \\
p'_i &= p_i + t\pi_i(x_1, \ldots, x_n, z, p_1, \ldots, p_n), \\
p'_{ij} &= p_{ij} + t\pi_{ij}(x_1, \ldots, x_n, z, p_1, \ldots, p_n, p_{11}, \ldots, p_{1n}, p_{21}, \ldots),
\end{aligned}$$

and so on, where in $\pi_{ij}, \ldots$ no derivatives of order higher than the second can occur, in $\pi_{ijk}, \ldots$ no derivatives of order higher than the third, and so generally.

We know how to express ζ, ξ_i, π_i, in terms of W and its derivatives, and we have now to express similarly $\pi_{ij}, \ldots$.

We have
$$dp'_k = \sum^{i=n} p'_{ki}\, dx'_i,$$
and therefore
$$d\pi_k = \sum^{i=n} \pi_{ki}\, dx_i + \sum^{i=n} p_{ki}\, d\xi_i,$$
so that
$$(1) \quad \sum^{i=n} \pi_{ki}\, dx_i - \sum^{j=i=n} p_{kij}\, \xi_i\, dx_i = d\left(\pi_k - \sum^{i=n} p_{ki}\, \xi_i\right).$$

If we use the symbol $\dfrac{d}{dx_k}$ to denote differentiation with respect to x_k, keeping $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ all constant, but not z or its derivatives, we have
$$\frac{d}{dx_k} = \frac{\partial}{\partial x_k} + p_k \frac{\partial}{\partial z} + \sum^{i=n} p_{ki} \frac{\partial}{\partial p_i} + \sum^{i=j=n} p_{kij} \frac{\partial}{\partial p_{ij}} + \ldots.$$

Now
$$-\pi_k = \frac{\partial W}{\partial x_k} + p_k \frac{\partial W}{\partial z}, \quad \xi_i = \frac{\partial W}{\partial p_i},$$
so that
$$\pi_k - \sum^{i=n} p_{ki}\, \xi_i = -\frac{\partial W}{\partial x_k} - p_k \frac{\partial W}{\partial z} - \sum^{i=n} p_{ki} \frac{\partial W}{\partial p_i} = -\frac{dW}{dx_k},$$
since W does not contain derivatives of order higher than the first.

From the equation (1) we can therefore deduce
$$\pi_{ki} - \sum^{j=n} p_{kji}\, \xi_j = -\frac{d^2 W}{dx_i\, dx_k}.$$

The result at which we have arrived may be thus stated:

$-\pi_i = \dfrac{dW}{dx_i}$, with the highest derivatives which occur omitted;

$-\pi_{ik} = \dfrac{d^2 W}{dx_i\, dx_k}$, with the highest derivatives which occur omitted.

In exactly the same manner we could prove that

$-\pi_{ijk} = \dfrac{d^3 W}{dx_i\, dx_j\, dx_k}$, with the highest derivatives omitted,

and so generally up to any assigned order; and we thus see how the infinitesimal contact transformation may be extended as far as we please.

The extended contact operator is
$$\sum^{i=n} \xi_i \frac{\partial}{\partial x_i} + \zeta \frac{\partial}{\partial z} + \sum^{i=n} \pi_i \frac{\partial}{\partial p_i} + \sum^{i=j=n} \pi_{ij} \frac{\partial}{\partial p_{ij}} + \ldots.$$

If we have a group of infinitesimal contact operators then these operators, when extended, will also form a group, of the same order as the original group, and with the same set of structure constants. This may be proved as in § 230, where a like theorem was proved for the point group extended, so as to be a contact group.

§ 236. It is convenient to have in explicit form the value of the first few coefficients in the operators for the case $n = 1$ and $n = 2$, as they are required for applications to geometry of two and three dimensions.

When $n = 1$, we take

$$W = p\xi - \eta,$$

and denote as usual

$$\frac{dy}{dx} \text{ by } p, \quad \frac{d^2y}{dx^2} \text{ by } q, \quad \frac{d^3y}{dx^3} \text{ by } r;$$

for $\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}$ we shall write X, and we now have

$$\xi = \frac{\partial W}{\partial p}, \quad \eta = p\frac{\partial W}{\partial p} - W, \quad \pi = -XW.$$

Also if $\quad q' = q + t\kappa$, and $r' = r + t\rho$,

we have $-\kappa = \frac{d^2 W}{dx^2}$, with the highest derivative omitted,

$$= \left(X + q\frac{\partial}{\partial p}\right)\left(X + q\frac{\partial}{\partial p}\right) W;$$

and therefore, since $\quad q\frac{\partial}{\partial p}X - qX\frac{\partial}{\partial p} = q\frac{\partial}{\partial y}$,

$$-\kappa = \left(X^2 + 2qX\frac{\partial}{\partial p} + q^2\frac{\partial^2}{\partial p^2} + q\frac{\partial}{\partial y}\right) W.$$

Similarly

$-\rho = \frac{d^3 W}{dx^3}$, with the highest derivative omitted,

$$= \left(X + q\frac{\partial}{\partial p} + r\frac{\partial}{\partial q}\right)\left(X + q\frac{\partial}{\partial p} + r\frac{\partial}{\partial q}\right)\left(X + q\frac{\partial}{\partial p}\right) W;$$

which, since $\quad \frac{\partial}{\partial p}X^2 - X^2\frac{\partial}{\partial p} = 2X\frac{\partial}{\partial y}$,

may be written

$$-\rho=\left(X^3+3qX^2\frac{\partial}{\partial p}+3q^2X\frac{\partial^2}{\partial p^2}+q^3\frac{\partial^3}{\partial p^3}+3qX\frac{\partial}{\partial y}+3q^2\frac{\partial^2}{\partial y\,\partial p}\right)W$$
$$+r\left(3q\frac{\partial^2}{\partial p^2}+3X\frac{\partial}{\partial p}+\frac{\partial}{\partial y}\right)W.$$

§ 237. As an example of the application of these formulae we shall find the form of those infinitesimal contact transformations which transform straight lines of the plane into straight lines.

The differential equation satisfied by all straight lines on the plane is $q=0$; and therefore, since we must have $q'=0$, we must have $\kappa=0$, wherever $q=0$. We therefore have $X^2W=0$; or, explicitly

$$(1) \qquad \frac{\partial^2 W}{\partial x^2}+2p\frac{\partial^2 W}{\partial x\,\partial y}+p^2\frac{\partial^2 W}{\partial y^2}=0,$$

of which the general integral is

$$W=f(y-px,\,p)+x\phi(y-px,\,p).$$

Any contact transformation, whose characteristic has this form, will transform any straight line into a straight line; these transformations have therefore the group property, but the group is not a finite one.

If W_1 and W_2 are two characteristic functions of this group the characteristic of the alternant of the operators $\overline{W}_1$ and $\overline{W}_2$ has, we know, the form $\{W_1, W_2\}$ where

$$\{W_1, W_2\}=(W_1, W_2)+W_1\frac{\partial W_2}{\partial y}-W_2\frac{\partial W_1}{\partial y},$$

and $$(W_1, W_2)=XW_1\frac{\partial W_2}{\partial p}-XW_2\frac{\partial W_1}{\partial p}.$$

We know then that W_1 and W_2 being any functional forms which satisfy the equation (1), $\{W_1, W_2\}$ will also be a functional form satisfying the same equation. This result may easily be verified independently.

If we only require those contact transformations which are mere extended point transformations, then by (1), since

$$W=p\xi-\eta,$$

and ξ and η do not now involve p,

$$p\xi_{11}-\eta_{11}+2p(p\xi_{12}-\eta_{12})+p^2(p\xi_{22}-\eta_{22})=0,$$

where the suffix 1 denotes differentiation with respect to x, and the suffix 2 differentiation with respect to y.

Equating to zero the coefficients of the several powers of p in this equation, we get

$$\xi_{22} = 0, \ \eta_{11} = 0, \ \eta_{22} - 2\xi_{12} = 0, \ \xi_{11} - 2\eta_{12} = 0.$$

Differentiating these equations with respect to x and y, we see that all derivatives of the third order are zero; we therefore take

$$\xi = a_1 x^2 + 2h_1 xy + b_1 y^2 + 2g_1 x + 2f_1 y + c_1,$$
$$\eta = a_2 x^2 + 2h_2 xy + b_2 y^2 + 2g_2 x + 2f_2 y + c_2.$$

From $\qquad \xi_{22} = \eta_{11} = 0$

we conclude that $\qquad a_2 = b_1 = 0$;

and from $\qquad \eta_{22} - 2\xi_{12} = 0$

we see that $\qquad 2h_1 - b_2 = 0$;

while from $\qquad \xi_{11} - 2\eta_{12} = 0$

we get $2h_2 = a_1$; and we thus obtain

$$W = a_1(px^2 - xy) + b_2(pxy - y^2) + 2g_1 px + 2f_1 py + c_1 p - 2g_2 x - 2f_2 y - c_2.$$

W is therefore merely the most general characteristic function of the extended projective group of the plane.

§ 238. We shall now find the form of those infinitesimal point transformations which have the property of transforming the circles of the plane into circles on the same plane.

The differential equation satisfied by all circles is

$$3q^2 p - (1 + p^2) r = 0,$$

and we must therefore have

$$(1) \qquad (1 + p^2)\rho + 2pr\pi - 6pq\kappa - 3q^2\pi = 0$$

for all values of x, y, p, q, r such that $3q^2 p = (1 + p^2) r$.

Since $\qquad W = p\xi - \eta,$

and the contact transformation is now a mere extended point transformation, W will only contain p in the first degree.

Applying the formulae of the preceding article to the equation (1), and substituting for r its equivalent expression in terms of p, q, we must have the equation

$$(1+p^2)\left(X^3+3qX^2\frac{\partial}{\partial p}+3q^2\frac{\partial^2}{\partial p\,\partial y}+3qX\frac{\partial}{\partial y}\right)W$$

$$(2)\qquad\qquad +3q^2p\left(3X\frac{\partial}{\partial p}+\frac{\partial}{\partial y}\right)W$$

$$=\left(3q^2-\frac{6p^2q^2}{1+p^2}\right)XW+6pq\left(X^2+2qX\frac{\partial}{\partial p}+q\frac{\partial}{\partial y}\right)W$$

satisfied for all values of x, y, p, q.

Equating the coefficients of q^2 on each side of this equation we have

$$(1+p^2)\,X\frac{\partial W}{\partial p}+p\left(3X\frac{\partial}{\partial p}+\frac{\partial}{\partial y}\right)W+\frac{p^2-1}{p^2+1}XW$$
$$=4pX\frac{\partial W}{\partial p}+2p\frac{\partial W}{\partial y}.$$

Substituting for W the expression $p\xi-\eta$, where ξ and η do not contain p, this is equivalent to

$$(1+p^2)^2\xi_2+(p^2-1)\left(p^2\xi_2+p(\xi_1-\eta_2)-\eta_1\right)$$
$$=p(1+p^2)(\xi_1+2p\xi_2-\eta_2).$$

Equating the coefficients of the different powers of p on each side we get the two equations

$$(3)\qquad \xi_1-\eta_2=0,\quad \xi_2+\eta_1=0.$$

Equating to zero the term in (2) which is independent of q, we get $X^3W=0$; that is,

$$p\xi_{111}+3p^2\xi_{112}+3p^3\xi_{122}+p^4\xi_{222}=\eta_{111}+3p\eta_{112}+3p^2\eta_{122}+p^3\eta_{222};$$

and therefore, since p, x, y are unconnected,

$$\eta_{111}=0,\ 3\eta_{112}-\xi_{111}=0,\ \eta_{122}-\xi_{112}=0,\ \eta_{222}-3\xi_{122}=0,\ \xi_{222}=0.$$

If we differentiate the equations (3) twice with respect to x and y, we shall see that all derivatives of ξ and η of the third order must be zero.

We therefore take

$$\xi=a_1x^2+2h_1xy+b_1y^2+2g_1x+2f_1y+c_1,$$
$$\eta=a_2x^2+2h_2xy+b_2y^2+2g_2x+2f_2y+c_2;$$

and from the equations (3) we deduce that

$$a_1=h_2,\ h_1=b_2,\ g_1=f_2,\ a_2+h_1=0,\ b_1+h_2=0,\ g_2+f_1=0,$$

so that the characteristic function is of the form

$$a_1\left(p(x^2-y^2)-2xy\right)+a_2(y^2-x^2-2pxy)+2g_1(px-y)$$
$$+2f_1(py+x)+c_1p-c_2.$$

It may at once be verified that for this value of W the coefficient of q vanishes in (2); and we thus see that there is a point group of order six which transforms circles into circles; the six independent operators of the group are

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},$$
$$(x^2-y^2)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}, \quad 2xy\frac{\partial}{\partial x}+(y^2-x^2)\frac{\partial}{\partial y}.$$

Of these infinitesimal operators the first corresponds geometrically to a small displacement along the axis of x; the second to a displacement along the axis of y; the third to a rotation round the origin; the fourth to a uniform expansion from the origin; the fifth to an inversion with respect to a circle of unit radius whose centre is the origin, succeeded by an inversion with respect to a circle of unit radius whose centre is at $x = t$, where t is small, and lastly, by a translation backwards along the axis of x measured by t; the sixth operator has a like interpretation with regard to the axis of y. It is of course obvious that each of these operations changes circles into neighbouring circles; and we have now proved that any infinitesimal transformation, which does so, must be compounded of these six operations.

§ 239. We next try whether there are any infinitesimal contact transformations—not mere extended point transformations—which have this property.

If we substitute in

$$(1+p^2)\rho+2pr\pi-6pq\kappa-3q^2\pi=0$$

for ρ, κ, π their values obtained in § 236; and then for r the expression $\dfrac{3pq^2}{1+p^2}$, the resulting equation must be satisfied for all values of x, y, p, q. Equating as before the coefficients of the different powers of q to zero, we obtain

$$(1+p^2)\frac{\partial^3 W}{\partial p^3}+3p\frac{\partial^2 W}{\partial p^2}=0,$$

$$(1+p^2)^2\left(X\frac{\partial^2 W}{\partial p^2}+\frac{\partial^2 W}{\partial p\partial y}\right)$$
$$+(p^2-1)XW-(p^2+1)\left(pX\frac{\partial W}{\partial p}+p\frac{\partial W}{\partial y}\right)=0,$$

$$(1+p^2)\left(X^2\frac{\partial W}{\partial p}+X\frac{\partial W}{\partial y}\right)-2pX^2W=0,$$

$$X^3W=0.$$

From the first of these equations we see that

$$(1+p^2)^{\frac{3}{2}}\frac{\partial^2 W}{\partial p^2}=A,$$

where A is a function of x and y only; and therefore

$$W=A\sqrt{1+p^2}+Bp+C,$$

where A, B, C are functions not containing p.

If this value of W is to satisfy the other equations it is clear from the irrationality of $\sqrt{1+p^2}$ that $A\sqrt{1+p^2}$ and $Bp+C$ must separately satisfy the equations. Now the latter part would give rise to a mere extended point transformation; and, as we have fully discussed all the point transformations which transform circles into circles, we need not further consider this part, but have only to find what, if any, are the possible values of the unknown function A.

Taking then $$W=A\sqrt{1+p^2},$$

we have $$\frac{\partial W}{\partial p}=\frac{Ap}{\sqrt{1+p^2}},$$

and the second equation gives us a mere identity satisfied whatever function A may be.

The third equation gives

$$(A_{11}+2A_{12}p+A_{22}p^2)\,p\sqrt{1+p^2}+(A_{12}+A_{22}p)\,(1+p^2)^{\frac{3}{2}}$$
$$=2p\sqrt{1+p^2}\,(A_{11}+2pA_{12}+p^2A_{22}),$$

which on dividing by $\sqrt{1+p^2}$ and equating the powers of p gives

(1) $$A_{11}=A_{22},\quad A_{12}=0.$$

Finally the fourth equation gives

$$A_{111}+3A_{112}p+3A_{122}p^2+A_{222}p^3=0,$$

from which we see that all derivatives of A above the second vanish; and therefore

$$A=ax^2+2hxy+by^2+2gx+2fy+c.$$

From (1) we further see that $h = 0$, and $a = b$, so that A is the power of a circle.

The most general contact transformation group which transforms circles into circles has therefore the following ten characteristic functions:

$$(2)\quad \begin{array}{l} (y^2+x^2)\sqrt{1+p^2},\ y\sqrt{1+p^2},\ x\sqrt{1+p^2},\ \sqrt{1+p^2}, \\ p(x^2-y^2)-2xy,\ y^2-x^2-2pxy,\ px-y,\ py+x,\ p,\ 1. \end{array}$$

§ 240. If we look on x, y, p as the coordinates of a point in three-dimensional space, to a point there will correspond an element of the plane; and to two united elements of the plane, that is, two consecutive elements whose coordinates satisfy the equation

$$dy-pdx = 0,$$

there will correspond two consecutive points in space connected by the equation

$$dy-pdx = 0.$$

If we write z for p we may say that to every transformation in space which leaves $dy-zdx = 0$ unaltered there corresponds a contact transformation in the plane, and conversely.

The group of contact transformations which we have just found leaves unaltered the system of circles

$$x^2+y^2+2gx+2fy+c = 0,$$

and therefore also

$$x+g+(y+f)p = 0.$$

The corresponding group of point transformations in three-dimensional space must therefore leave unaltered the system of curves given by

$$x^2+y^2+2gx+2fy+c = 0,\quad x+g+(y+f)z = 0;$$

that is, will transform any curve of this system into some other curve of the same system.

It is now convenient to write the equations of this family of curves in the form

$$(1)\quad \begin{array}{l} 4c(x^2+y^2)+4(b^2-ac)(y+ix)+y-ix-a = 0, \\ 8c(x+yz)+4(b^2-ac)(z+c)+z-i = 0, \end{array}$$

where a, b, c are variable parameters, and i is the symbol for $\sqrt{-1}$.

If we apply the transformation

$$(2) \qquad x' = y + ix, \quad y' = y - ix, \quad z' = \frac{z-i}{z+i},$$

which leaves unaltered the equation

$$dy - zdx = 0,$$

the equations (1) are transformed to

$$(3) \qquad \begin{aligned} 4cxy + 4(b^2 - ac)x + y - a &= 0, \\ 4c(y + xz) + 4(b^2 - ac) + z &= 0; \end{aligned}$$

so that the group into which the group (2) of § 239 is transformed by the equations (2) of the present article leaves the equations (3) unaltered.

Transform again with

$$y = y' - \tfrac{1}{2}x'z', \quad x = -\tfrac{1}{2}\frac{x'}{z'}, \quad z = -z'^2,$$

which gives $\qquad dy - zdx = dy' - z'dx'$;

and the equations (3) become transformed into

$$-2cxy + cx^2z - 2(b^2 - ac)x + yz - \tfrac{1}{2}xz^2 - az = 0,$$
$$4cy + 4(b^2 - ac) - z^2 = 0.$$

Eliminating z between these two equations we get

$$(cx^2 - y + a)^2 = 4b^2x^2;$$

and therefore, since b is a variable parameter, we may write these equations in the form

$$(4) \qquad y = cx^2 + 2bx + a, \quad z = 2b + 2cx.$$

The group into which the group (2) of § 239 is now transformed leaves the system (4) unaltered; or, expressed as a contact group in the plane, leaves invariant the system of parabolas whose axes are parallel to the fixed line $x = 0$; or, again, leaves unaltered the differential equation

$$(5) \qquad \frac{d^3y}{dx^3} = 0.$$

The group into which (2) of § 239 is transformed could have been directly obtained from this property of leaving (5) unaltered, just as (though more simply than) the group which left circles unaltered was obtained. If the group is thus directly obtained, it will serve as an example of the applica-

tion of § 231, to prove that the two groups are transformable, the one into the other, by a contact transformation.

§ 241. Let us next apply the point transformation in three-dimensional space

$$x' = x, \quad y' = y - \tfrac{1}{2}xz, \quad z' = \tfrac{1}{2}z,$$

for which $\qquad dy - zdx = dy' - z'dx' + x'dz',$

and for which therefore a linear element of any curve in the plane is transformed into a linear element of the linear complex $m = \beta$.

We then see that the group of contact transformations, which leaves unaltered the system of parabolas, is transformed into a group of point transformations in three-dimensional space, with the property of leaving unaltered the system of straight lines

$$y = bx + a, \quad z = cx + b;$$

that is, into a projective group which does not alter the linear complex $m = \beta$.

We have thus established a correspondence between the circles of a plane, and the straight lines of a linear complex in space of three dimensions; and the two groups, one a contact transformation group in x, y, p, leaving the system of circles unaltered, and the other a point group which transforms the straight lines of a given linear complex *inter se*, are transformable, the one into the other, by a point transformation in three-dimensional space. It should be noticed, however, that this point transformation is not a contact transformation in x, y, p, such as was that which transformed the system of circles into a system of parabolas.

If we write the equation of a circle in the plane in the form

$$(x-\alpha)^2 + (y-\beta)^2 + \gamma^2 = 0,$$

then the group of transformations, which transform any one circle into any other, being a contact group, will transform two circles which touch into two other circles which touch.

Now we have seen, in Chapter VIII, that if a group transforms an equation of the form

$$f(x_1, \ldots, x_n, a_1, \ldots, a_r) = 0$$

into another equation of like form, but with a different set of parameters, then we can construct a group of transformations in the variables $a_1, \ldots, a_r$, such that if $X_1, \ldots, X_m$ are the operators of the group in the letters $x_1, \ldots, x_n$ and $A_1, \ldots, A_m$

the operators in the letters $a_1, \ldots, a_r$, the structure constants of the two will be the same; and each of the operators

$$A_1+X_1, \ldots, A_m+X_m$$

will be admitted by the equation

$$f(x_1, \ldots, x_n, a_1, \ldots, a_r) = 0.$$

If we apply this method to the system of circles on the plane which admit the group (2) of § 239, we shall have a group in the variables α, β, γ; this group will be of the tenth order, and will be found to be the group of conformal transformations in three-dimensional space.

This result is obtained directly by Lie from the consideration that the condition for two neighbouring circles touching is

$$d\alpha^2+d\beta^2+d\gamma^2 = 0;$$

for, since the transformed neighbouring circles must also touch, the equation

$$d\alpha^2+d\beta^2+d\gamma^2 = 0$$

must be unaltered; that is, the group must be the conformal one.

§ 242. We shall now write down in explicit form (for the case $n = 2$) the values of the functions $\pi_{11}, \pi_{12}, \pi_{22}$ which in future we shall denote by ρ, σ, τ.

We have (p, q, r, s, t having their usual meaning)

$$W = p\xi+q\eta-\zeta,$$

and the infinitesimal operator is

$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}+\pi\frac{\partial}{\partial p}+\kappa\frac{\partial}{\partial q}+\rho\frac{\partial}{\partial r}+\sigma\frac{\partial}{\partial s}+\tau\frac{\partial}{\partial t}.$$

We denote by X and Y the respective operators

$$\frac{\partial}{\partial x}+p\frac{\partial}{\partial z} \text{ and } \frac{\partial}{\partial y}+q\frac{\partial}{\partial z};$$

and we have

$$\xi=\frac{\partial W}{\partial p}, \quad \eta=\frac{\partial W}{\partial q}, \quad \zeta=p\frac{\partial W}{\partial p}+q\frac{\partial W}{\partial q}-W,$$

$$-\pi=\frac{\partial W}{\partial x}+p\frac{\partial W}{\partial z}=XW, \quad -\kappa=\frac{\partial W}{\partial y}+q\frac{\partial W}{\partial z}=YW.$$

Since $-\rho = \dfrac{d^2 W}{dx^2}$, with the highest derivatives omitted,

$$= \left(X + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q}\right)\left(X + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q}\right) W,$$

and since $$\frac{\partial}{\partial p} X - X\frac{\partial}{\partial p} = \frac{\partial}{\partial z} = \frac{\partial}{\partial q} Y - Y\frac{\partial}{\partial q},$$

$$\frac{\partial}{\partial p} Y - Y\frac{\partial}{\partial p} = 0, \quad \frac{\partial}{\partial q} X - X\frac{\partial}{\partial q} = 0,$$

we have

$$-\rho = \left(X^2 + 2rX\frac{\partial}{\partial p} + 2sX\frac{\partial}{\partial q} + r^2\frac{\partial^2}{\partial p^2} + 2rs\frac{\partial^2}{\partial p\,\partial q} + s^2\frac{\partial^2}{\partial q^2} + r\frac{\partial}{\partial z}\right) W.$$

Similarly $-\sigma$ and $-\tau$ are obtained from the operators

$$XY + sX\frac{\partial}{\partial p} + tX\frac{\partial}{\partial q} + rY\frac{\partial}{\partial p} + sY\frac{\partial}{\partial q} + rs\frac{\partial^2}{\partial p^2} + (rt + s^2)\frac{\partial^2}{\partial p\,\partial q} + st\frac{\partial^2}{\partial q^2} + s\frac{\partial}{\partial z}$$

and $$Y^2 + 2sY\frac{\partial}{\partial p} + 2tY\frac{\partial}{\partial q} + s^2\frac{\partial^2}{\partial p^2} + 2st\frac{\partial^2}{\partial p\,\partial q} + t^2\frac{\partial^2}{\partial q^2} + t\frac{\partial}{\partial z}.$$

§ 243. As an example of the application of these formulae we shall find the form of the most general infinitesimal contact transformation which does not alter

$$\frac{d^2 z}{dx\,dy} = 0.$$

Since we must have $\sigma = 0$ wherever $s = 0$, we get

$$\frac{\partial^2 W}{\partial p\,\partial q} = 0, \quad X\frac{\partial W}{\partial q} = 0, \quad Y\frac{\partial W}{\partial p} = 0, \quad XYW = 0.$$

From the first of these equations we see that $\dfrac{\partial W}{\partial q}$ does not contain p; and therefore by the second we must have

$$\frac{\partial^2 W}{\partial q\,\partial z} = 0, \quad \frac{\partial^2 W}{\partial q\,\partial x} = 0,$$

so that $\dfrac{\partial W}{\partial q}$ is a function of y and q only. Similarly we see that $\dfrac{\partial W}{\partial p}$ is a function of x and p only, and therefore the characteristic function W is of the form

$$f(p, x) + \phi(q, y) + \psi(x, y, z).$$

Since XYW vanishes identically,

$$\psi_{12}+p\psi_{23}+q\psi_{13}+pq\psi_{33} = 0\,;$$

and therefore

$$\psi_{12} = 0, \quad \psi_{23} = 0, \quad \psi_{13} = 0, \quad \psi_{33} = 0,$$

so that
$$\psi(x, y, z) = az + F(x) + \Phi(y),$$

where a is a mere constant and F and Φ functional forms.

The characteristic function which leaves unaltered the equation $s = 0$ is therefore of the form

$$f(p, x) + \phi(q, y) + az.$$

There are therefore three distinct forms of characteristic functions leaving $s = 0$ unaltered; and, corresponding to these, three distinct groups of contact transformations with this property. Firstly, the infinite group where W is of the form $f(p, x)$, f being an arbitrary functional symbol; the functions of this group form a function system of the second order. Secondly, the infinite group with characteristic functions of the form $\phi(q, y)$, where ϕ is an arbitrary functional symbol; the functions of this system also form a function system of the second order, any function of which is in involution with any function of the first system. Thirdly, the group with the single characteristic function z; if we form the alternant of this function with any function of the first system, we have another function of the first system; and a similar result follows for the alternant of z with any function of the second system.

The infinite group of contact transformations leaving unaltered the equation $s = 0$ is compounded of the operations of these three groups.

We have proved that any Ampèrian equation with intermediary integrals of the form

$$u_1 = f_1(v_1) \quad \text{and} \quad u_2 = f_2(v_2),$$

where f_1 and f_2 are arbitrary functional forms, can by a contact transformation be reduced to the form $s = 0$.

It follows that any such Ampèrian equation will admit an infinite group of infinitesimal contact transformations, the operators of which may be arranged in classes as follows: in the first class there are two *unconnected* operators, but an infinite number of *independent* operators: in the second class there are also two unconnected operators, and an infinite number of independent operators: in the third class there is only one operator: any operator of the first class is

permutable with any of the second, and the alternant of the operator of the third class with any operator of one of the other classes is an operator of that other class.

§ 244. We have obtained the conformal group in three-dimensional space from the property that it leaves the equation

$$dx^2+dy^2+dz^2 = 0$$

unaltered; if we seek the group which will leave the expression

$$dx^2+dy^2+dz^2$$

unaltered, we shall obtain the group of movements of a rigid body.

The question now proposed is to find the infinitesimal point transformations which have the property of transforming a given surface into a neighbouring one, without altering the length of arcs on the surface; that is, if P and Q are any two neighbouring points on a given surface which receive infinitesimal displacements so as to become two near points P', Q' on a neighbouring surface, we want to find the relations between ξ, η, ζ in order that we may have $PQ = P'Q'$.

Since we must have

$$dx\,d\xi+dy\,d\eta+dz\,d\zeta = 0$$

for all values of x, y, z on the given surface; and

$$d\xi = \frac{d\xi}{dx}dx + \frac{d\xi}{dy}dy, \quad d\eta = \frac{d\eta}{dx}dx + \frac{d\eta}{dy}dy,$$

$$d\zeta = \frac{d\zeta}{dx}dx + \frac{d\zeta}{dy}dy, \quad dz = pdx+qdy,$$

we get, by equating the coefficients of dx^2, $dxdy$, dy^2 to zero,

$$(1) \quad \frac{d\xi}{dx}+p\frac{d\zeta}{dx} = 0, \quad \frac{d\xi}{dy}+\frac{d\eta}{dx}+p\frac{d\zeta}{dy}+q\frac{d\zeta}{dx} = 0,$$

$$\frac{d\eta}{dy}+q\frac{d\zeta}{dy} = 0,$$

where $\frac{d}{dx}$ and $\frac{d}{dy}$ denote total differentiation with respect to x and to y.

From the equations

$$\frac{d^2}{dxdy}\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}+p\frac{d\zeta}{dy}+q\frac{d\zeta}{dx}\right) = 0,$$

$$\frac{d^2}{dy^2}\left(\frac{d\xi}{dx}+p\frac{d\zeta}{dx}\right) = 0, \quad \frac{d^2}{dx^2}\left(\frac{d\eta}{dy}+q\frac{d\zeta}{dy}\right) = 0$$

we can eliminate ξ and η, and thus obtain the equation

$$(2) \qquad t\frac{d^2\zeta}{dx^2} - 2s\frac{d^2\zeta}{dx\,dy} + r\frac{d^2\zeta}{dy^2} = 0.$$

The surface on which P and Q lie is a known one, and therefore r, s, t are known in terms of x, y, so that the equation (2) determines ζ as a function of x and y.

From

$$\frac{d}{dy}\left(\frac{d\xi}{dy} + \frac{d\eta}{dx} + p\frac{d\zeta}{dy} + q\frac{d\zeta}{dx}\right) = 0, \quad \frac{d}{dx}\left(\frac{d\eta}{dy} + q\frac{d\zeta}{dy}\right) = 0$$

we get
$$\frac{d^2\xi}{dy^2} + p\frac{d^2\zeta}{dy^2} + t\frac{d\zeta}{dx} = 0;$$

while by differentiating

$$\frac{d\xi}{dx} + p\frac{d\zeta}{dx} = 0$$

with respect to x and with respect to y we get

$$\frac{d^2\xi}{dx^2} + p\frac{d^2\zeta}{dx^2} + r\frac{d\zeta}{dx} = 0, \text{ and } \frac{d^2\xi}{dx\,dy} + p\frac{d^2\zeta}{dx\,dy} + s\frac{d\zeta}{dx} = 0,$$

with similar equations for η.

If we denote $\frac{d\eta}{dx} - \frac{d\xi}{dy}$ by λ

we have, therefore,

$$d\lambda = \left(p\frac{d^2\zeta}{dx\,dy} - q\frac{d^2\zeta}{dx^2} + s\frac{d\zeta}{dx} - r\frac{d\zeta}{dy}\right)dx$$
$$+ \left(p\frac{d^2\zeta}{dy^2} - q\frac{d^2\zeta}{dx\,dy} + t\frac{d^2\zeta}{dx\,dy} - s\frac{d\zeta}{dy}\right)dy,$$

which is a perfect differential, since

$$t\frac{d^2\zeta}{dx^2} - 2s\frac{d^2\zeta}{dx\,dy} + r\frac{d^2\zeta}{dx^2} = 0;$$

and therefore λ can be obtained by quadratures, when ζ is known in terms of x, y.

When we know λ and ζ, the derivatives of ξ and η are known by (1); and therefore ξ and η can be obtained by quadratures. It will also be noticed that when ζ is fixed, ξ and η are fixed, save as to the terms $ay+b$ in ξ and $-ax+c$

in η where a, b, c are arbitrary constants. The infinitesimal transformation is therefore fixed when ζ is fixed, except for small translations along the axes of x and y, and rotations round the axis of z.

The mistake of supposing that the operators

$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}$$

so found will generate a group must be guarded against: if U is an operation which transforms a surface S into Σ and preserves unaltered the lengths of small arcs on S, and V is another operation with the same property, then VU will not necessarily have the required property, because V has not necessarily such a property for the surface Σ.

§ 245. We can now employ the values of ρ, σ, τ obtained in § 242 to prove the known theorem, that any such infinitesimal transformation as we are now considering will so transform S into Σ, that the measure of curvature will be the same at corresponding points on these surfaces.

We have

$$-t\rho-r\tau+2s\sigma=\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)W,$$

omitting derivatives of the highest order which occur, that is, derivatives of the third order; and this expression is equal to

$$\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)(p\xi+q\eta)$$

since $$\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)\zeta=0.$$

Now

$$\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)p\xi=p\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)\xi$$
$$+2tr\frac{d\xi}{dx}+2sr\frac{d\xi}{dy}-2sr\frac{d\xi}{dy}-2s^2\frac{d\xi}{dx},$$

the other terms being omitted as they are derivatives of the third order.

If we now make use of the equations (1) of § 244 to express the derivatives of ξ of the second order in terms of those of ζ, we have

$$\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)p\xi=-p^2\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)\zeta$$
$$-2p(rt-s^2)\frac{d\zeta}{dx}+2(rt-s^2)\frac{d\xi}{dx}$$
$$=2(rt-s^2)\left(\frac{d\xi}{dx}-p\frac{d\zeta}{dx}\right)=-4(rt-s^2)p\frac{d\zeta}{dx}$$

by (1) of § 244.

Similarly we see that

$$\left(t\frac{d^2}{dx^2}+r\frac{d^2}{dy^2}-2s\frac{d^2}{dxdy}\right)q\eta=-4(rt-s^2)q\frac{d\zeta}{dy}.$$

Again

$$-p\pi-q\kappa=\left(p\frac{d}{dx}+q\frac{d}{dy}\right)(p\xi+q\eta-\zeta),$$ omitting the highest derivatives

$$=p^2\frac{d\xi}{dx}+q^2\frac{d\eta}{dy}+pq\left(\frac{d\xi}{dy}+\frac{d\eta}{dx}\right)-p\frac{d\zeta}{dx}-q\frac{d\zeta}{dy}$$
$$=-(1+p^2+q^2)\left(p\frac{d\zeta}{dx}+q\frac{d\zeta}{dy}\right)$$ by (1) of § 244.

Now in order to prove that the measure of curvature is unaltered by the given infinitesimal transformations, it is only necessary to prove that

$$(1+p^2+q^2)(t\rho+r\tau-2s\sigma)=4(rt-s^2)(p\pi+q\kappa);$$

and this is at once proved by aid of the formulae now obtained.

§ 246. If we have an ∞^2 of points on a surface and the distance between neighbouring points (measured along a geodesic on the surface) is invariable as this ∞^2 of points moves on the surface, we then have on the surface the analogue of a rigid lamina in a plane. Such an assemblage we call a *net*; and the question is suggested, can a movable net exist on any surface, or can it only exist on particular classes of surfaces?

If P is any point on the net which moves to a neighbouring point P', we have just proved that the measure of curvature at P and P' must be the same; we shall first discuss the case where the given surface has not everywhere the same measure of curvature.

Through each point on the surface draw the curve along which the measure of curvature is constant, and let these

curves be called the curves of constant curvature. Next draw the system of curves cutting these curves of constant curvature orthogonally, and call these latter curves the trajectories.

Let $A_1, A_2, \ldots$ be a series of neighbouring points on a trajectory; if the set is movable $A_1, A_2, \ldots$ will take up positions $B_1, B_2, \ldots$ and the points of the net which were at $B_1, B_2, \ldots$ originally will now take up a position $C_1, C_2, \ldots$ and so on.

The points $A_1, B_1, C_1, \ldots$ must lie on a line of constant curvature; similarly $A_2, B_2, C_2, \ldots$ must lie on such a line, $A_3, B_3, C_3, \ldots$ on another, and so on. It will now be proved that this net movement is only possible if $B_1, B_2, \ldots$ lie on a trajectory, $C_1, C_2, \ldots$ also on a trajectory, and so on.

Since $A_1 B_1 = B_1 C_1$ and $A_1 A_2 = B_1 B_2$ and $A_2 B_1 = B_2 C_1$, it follows that the angle $A_2 A_1 B_1 = B_2 B_1 C_1$; and therefore, since $A_2 A_1 B_1$ is a right angle, so is $B_2 B_1 C_1$; that is, $B_1, B_2, \ldots$ lie on a trajectory.

Unless then the surface is such that trajectories can be drawn on it, dividing each line, along which the measure of curvature is constant, into the same number of equal parts, the surface cannot allow a net to move over it. If this condition is satisfied, and the surface be not one with the same measure of curvature everywhere, the net can move on it with one, and only one, degree of freedom.

Since $A_1 A_2 = B_1 B_2$ the perpendicular distance between two neighbouring lines of constant curvature is the same at all points; it therefore follows that the trajectories are geodesics on the surface.

If we take u and v to be the coordinates of any point on the surface, where $u = \alpha$ and $v = \beta$ are respectively the lines of constant curvature and their trajectories, we can take for the element of length on the surface

$$ds^2 = du^2 + \lambda^2 dv^2$$

when λ is a function of u only.

If the net is to have two degrees of freedom in its movements the surface must be everywhere of the same measure of curvature.

§ 247. We can prove these results in a different manner and also obtain all possible movements of the net, if we employ surface coordinates.

Let the equation of the surface be given in the form

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

so that we have

$$ds^2 = edu^2 + 2fdudv + gdv^2,$$

where e, f, g are functions of the parameters u, v which define the position of any point on the surface.

We shall first prove that by proper choice of the parameters we may take $e = 1, f = 0$, and thus simplify the expression for the element of length.

We must prove that we can find p and q, a pair of functions of u and v such that

$$edu^2 + 2fdudv + gdv^2 = dp^2 + \lambda^2 dq^2.$$

Since

$$dp = \frac{\partial p}{\partial u}du + \frac{\partial p}{\partial v}dv \text{ and } dq = \frac{\partial q}{\partial u}du + \frac{\partial q}{\partial v}dv,$$

we at once obtain as the necessary and sufficient conditions for such reduction

$$e = \left(\frac{\partial p}{\partial u}\right)^2 + \lambda^2\left(\frac{\partial q}{\partial u}\right)^2, \quad f = \frac{\partial p}{\partial u}\frac{\partial p}{\partial v} + \lambda^2\frac{\partial q}{\partial u}\frac{\partial q}{\partial v},$$

$$g = \left(\frac{\partial p}{\partial v}\right)^2 + \lambda^2\left(\frac{\partial q}{dv}\right)^2;$$

and therefore

$$\left(e - \left(\frac{\partial p}{\partial u}\right)^2\right)\left(g - \left(\frac{\partial p}{\partial v}\right)^2\right) = \left(f - \frac{\partial p}{\partial u}\frac{\partial p}{\partial v}\right)^2.$$

It follows that p must satisfy the equation

$$eg - f^2 = g\left(\frac{\partial p}{\partial u}\right)^2 - 2f\frac{\partial p}{\partial u}\frac{\partial p}{\partial v} + e\left(\frac{\partial p}{\partial v}\right)^2.$$

When we have thus determined p as a function of u and v, we can determine λ and q by the equations

$$\lambda\frac{\partial q}{\partial u} = \sqrt{e - \left(\frac{\partial p}{\partial u}\right)^2}, \quad \lambda\frac{\partial q}{\partial v} = \sqrt{g - \left(\frac{\partial p}{\partial v}\right)^2};$$

eliminating q we have, for determining λ, the equation

$$\frac{\partial}{\partial v}\lambda^{-1}\sqrt{e - \left(\frac{\partial p}{\partial u}\right)^2} = \frac{\partial}{\partial u}\lambda^{-1}\sqrt{g - \left(\frac{\partial p}{\partial v}\right)^2}.$$

When λ is thus determined we can find q by quadratures. We have therefore proved the theorem we stated, viz. that by

a suitable choice of surface coordinates we may take

(1) $$ds^2 = dp^2 + \lambda^2 dq^2.$$

If we form the differential equation of the geodesics on the surface with respect to this system of coordinates, we shall see that it is satisfied by the curves $q =$ constant: these curves are therefore geodesics.

§ 248. We can throw this expression into another form which will also be required in our investigation; take a new set of parameters such that

$$dp + i\lambda dq = \mu d\alpha \text{ and } dp - i\lambda dq = \nu d\beta,$$

where i is the symbol for $\sqrt{-1}$; that is, $\frac{1}{\mu}$ is the integrating factor of $dp + i\lambda dq$ and $\frac{1}{\nu}$ the corresponding factor for $dp - i\lambda dq$; we now have

$$ds^2 = e^h\, d\alpha\, d\beta,$$

where h is some function of α and β.

It is convenient to write x for α and y for β so that

$$ds^2 = e^h\, dx\, dy.$$

Suppose now that points on the surface admit the infinitesimal transformation

$$x' = x + t\xi(x, y), \quad y' = y + t\eta(x, y),$$

which does not alter the length of arcs; that is, suppose that a movable net can exist on the surface.

Since ds is to be unaltered we must have

$$dx\, d\eta + dy\, d\xi + dx dy \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) h = 0;$$

and therefore by equating the coefficients of dx^2, $dxdy$, dy^2 to zero we get

$$\frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) h = 0.$$

From these equations we conclude that ξ is a function of x only, and η a function of y only; and therefore, by taking as parameters, instead of x, a suitable function of x, and, instead of y, a suitable function of y, we may in the new coordinates take ξ and η each to be unity. In fact if $\xi = f(x)$

then from $$x' = x + t f(x),$$

we conclude that whatever ϕ may be,

$$\phi(x') = \phi(x) + t f(x) \phi'(x);$$

if then we take $\phi'(x) f(x)$ to be unity, and $\phi(x)$ as a new parameter in place of x, ξ will be unity.

Since we must now have with these parameters

$$\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} = 0,$$

h must be a function of $x-y$.

We can therefore, if the surface can have a movable net drawn on it, so choose our surface coordinates that

$$e^h + 4\left(\frac{\partial f(x-y)}{\partial x}\right)^2 = 0,$$

where f is some functional symbol; and we have

$$ds^2 = -\left(\frac{\partial}{\partial x} f(x-y)\right)^2 \left((dx+dy)^2 - (dx-dy)^2\right)$$

$$= (df)^2 - \left(\frac{\partial f}{\partial x}\right)^2 (dx+dy)^2.$$

This form is the same as (1) of § 247, only that λ^2 is now a function of p only and not of q; and we conclude that the net can move, if and only if, the element of arc can be written

in the form $$ds^2 = dp^2 + \lambda^2 dq^2$$

where λ^2 is a function of p only.

§ 249. We now assume the surface to be such that we may take $$ds^2 = dx^2 + \lambda^2 dy^2$$

where λ is a function of x only.

It is known (Salmon, *Geometry of Three Dimensions*, § 389) that the measure of curvature is $\dfrac{d^2\lambda}{dx^2} \div \lambda$; and therefore the lines on the surface where the measure of curvature is constant are the lines $x =$ constant.

To find the most general displacement of the net on the surface we now have

$$dx\,d\xi + \lambda^2 dy\,d\eta + dy^2\left(\xi \frac{\partial \lambda^2}{\partial x} + \eta \frac{\partial \lambda^2}{\partial y}\right);$$

and therefore, since λ does not contain y,

$$(1)\ \frac{\partial\xi}{\partial x}=0,\quad (2)\ \lambda^2\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}=0,\quad (3)\ \lambda^2\frac{\partial\eta}{\partial y}+\xi\frac{\partial\lambda^2}{\partial x}=0.$$

Eliminating η from the second and third of these equations we get

$$\frac{\partial}{\partial y}\Big(\frac{1}{\lambda^2}\frac{\partial\xi}{\partial y}\Big)=\frac{\partial}{\partial x}\Big(\frac{\xi}{\lambda^2}\frac{\partial\lambda^2}{\partial x}\Big);$$

and therefore

$$\frac{\partial^2\xi}{\partial y^2}=2\xi\Big(\lambda\frac{\partial^2\lambda}{\partial x^2}-\Big(\frac{\partial\lambda}{\partial x}\Big)^2\Big).$$

From the first equation we see that ξ is a function of y only.
First suppose that ξ is zero, then

$$\frac{\partial\eta}{\partial x}=0,\quad \frac{\partial\eta}{\partial y}=0;$$

and we get the possible displacement

$$x'=x,\quad y'=y+t;$$

that is, a displacement along a line where the measure of curvature is constant.

If ξ is not zero, since

$$\frac{\partial^2\xi}{\partial y^2}\div\xi=2\Big(\lambda\frac{\partial^2\lambda}{\partial x^2}-\Big(\frac{\partial\lambda}{\partial x}\Big)^2\Big),$$

and ξ is a function of y, and λ a function of x, each of these equal expressions must be a mere constant.

Suppose that this constant is not zero, then

$$\lambda\frac{d^2\lambda}{dx^2}-\Big(\frac{d\lambda}{dx}\Big)^2=a^2.$$

Solving this equation we get

$$\lambda=\frac{a}{k}\cosh(kx+\epsilon),$$

where ϵ and k are constants; and this value of λ gives the measure of curvature constant everywhere on the surface, and equal to k^2.

From $$\frac{d^2\xi}{dy^2}=2a^2\xi,$$

we get $$\xi=A\cosh\sqrt{2}ay+B\sinh\sqrt{2}ay;$$

and from (2) and (3) we now have

$$\eta = -\frac{\sqrt{2}\,k}{a}\tanh(kx+\epsilon)\,(A\sinh\sqrt{2}\,ay + B\cosh\sqrt{2}\,ay) + C,$$

where A, B, C are arbitrary constants.

If we take $\lambda\dfrac{d^2\lambda}{dx^2} - \left(\dfrac{d\lambda}{dx}\right)^2$ to be negative and equal to $-a^2$,

we should take λ to be $\dfrac{a}{k}\cos(kx+\epsilon)$, and

$$\xi = A\cos\sqrt{2}\,ay + B\sin\sqrt{2}\,ay,$$

$$\eta = \frac{\sqrt{2}\,k}{a}\tan(kx+\epsilon)\,(A\sin\sqrt{2}\,ay - B\cos\sqrt{2}\,ay) + C;$$

the measure of curvature at any point of the surface is then equal to $-k^2$.

By properly choosing the initial line from which x is to be measured we may take ϵ to be $\dfrac{\pi}{2}$ when λ becomes $-\dfrac{a}{k}\sin kx$.

In particular when k is zero, that is, when the surface is a developable,

$$\lambda = -ax, \quad \xi = A\cos\sqrt{2}\,ay + B\sin\sqrt{2}\,ay,$$

$$\eta = -\frac{\sqrt{2}}{ax}(A\sin\sqrt{2}\,ay - B\cos\sqrt{2}\,ay) + C.$$

In general, then, we have three linear operators corresponding to the three possible infinitesimal displacements of the net; and for the case where $\lambda\dfrac{d^2\lambda}{dx^2} - \left(\dfrac{d\lambda}{dx}\right)^2$ is negative and not zero these operators are X_1, X_2, X_3 where

$$X_1 = \cos\sqrt{2}\,ay\frac{\partial}{\partial x} - \sqrt{2}\frac{k}{a}\cot kx\sin\sqrt{2}\,ay\frac{\partial}{\partial y},$$

$$X_2 = \sin\sqrt{2}\,ay\frac{\partial}{\partial x} + \sqrt{2}\frac{k}{a}\cot kx\cos\sqrt{2}\,ay\frac{\partial}{\partial y},$$

$$X_3 = \frac{\partial}{\partial y}.$$

We obtain by simple calculation

$$(X_3, X_1) = -\sqrt{2}\,aX_2, \quad (X_2, X_3) = -\sqrt{2}\,aX_1,$$

$$(X_1, X_2) = \frac{-2\sqrt{2}\,k^2}{a}X_3.$$

The discussion of the case where $\lambda \dfrac{d^2\lambda}{dx^2} - \left(\dfrac{d\lambda}{dx}\right)^2$ is zero may be left to the reader; it need only be stated that it cannot be deduced from the results given by merely taking a to be zero.

The general result of this discussion is therefore to show that, if a surface is not one over which the measure of curvature is everywhere the same, at the most there can be but one degree of freedom in the motion of the net; and also that no movement of the net is possible at all, unless the surface is such that the perpendicular distance between any two neighbouring lines, along each of which the measure of curvature is constant, is the same at all points of the line.

On surfaces, however, with a constant measure of curvature the net can move with three degrees of freedom; and the movements of the net generate a group of the third order. This group will contain a pair of permutable operators if the surface is a developable.

CHAPTER XX

DIFFERENTIAL INVARIANTS

§ 250. If we are given any function of $z, x_1, \ldots, x_n$ we know that there are n unconnected linear operators which will annihilate the function; these operators form a group, though not necessarily a finite group, with respect to which the given function is invariant: and more generally, if we are given m such functions of the variables $f_1, \ldots, f_m$, there will be $(n+1-m)$ unconnected operators forming a group, with respect to which $f_1, \ldots, f_m$ will be invariants.

So too when we are given a linear partial differential equation of the first order, or a complete system of such equations, we have seen in Chapter VII how the system must admit a complete system of linear operators generating a group. If the system of equations is of the first order, but not linear, then, though it will not generally admit any group of point transformations, yet it will admit a group of contact transformations. In particular cases the equations when not linear may admit groups of point transformations; thus we found (§§ 33–35) that the equation

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 0,$$

admitted the conformal group of three-dimensional space.

In general, differential equations of order above the first do not admit point transformation groups, but some particular equations do; thus

$$\frac{d^2 y}{dx^2} = 0$$

admits the projective group of the plane; the expression

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \div \frac{d^2 y}{dx^2},$$

for the radius of curvature admits the group of movements of a rigid lamina in the plane. If we are given any differential

expression or equation, we have seen in Chapter XIX how to determine the infinitesimal point transformations which it may admit; we have also considered examples of determining the transformations admitted by equations of the form

$$f(z, x_1, \ldots, x_n, dx_1, \ldots, dx_n) = 0,$$

and we have seen how closely all these different problems are connected with the idea of extended point groups. The method common to the solution of these problems was that of determining the group admitted by a given expression (or equation) which expression is then an invariant of the group; that is, the invariant was given, and the group was then to be found.

§ 251. In this chapter we shall consider the converse problem, viz. how, when the group is given, we are to obtain the functions of $z, x_1, \ldots, x_n$, and the derivatives of z, which preserve their form under all the operations of the group; in other words, we are to investigate how the differential invariants of known groups are to be calculated. We confine ourselves to the case where the group is a finite continuous one.

We have solved a part of the proposed problem in Chapter VIII, where we showed how to obtain the functions of $z, x_1, \ldots, x_n$ which are invariant for a known group, and also how to find all the equations which the group admits. Such functions, or equations, may be considered as respectively differential invariants of zero order or differential equations of zero order; and we have seen that only intransitive groups can have differential invariants of zero order, whilst imprimitive groups must have an invariant system of differential equations of the first order.

Suppose that we now wish to find all the differential invariants of the k^{th} order of a known group, that is, invariants involving derivatives of the k^{th} order. We first extend the operators of the group to the k^{th} order, when we shall have the operators of a group in the variables $z, x_1, \ldots, x_n$, and the derivatives of z up to the k^{th} order; this group has the same structure constants as the given group in $z, x_1, \ldots, x_n$.

We then apply the general method to this extended group, and find its differential invariants of zero order, and these will be differential invariants of the original group involving the k^{th} derivatives of z; that is, they will be what we have called invariants of the k^{th} order.

In exactly the same manner, we see how the problem of finding the invariant differential equations of the k^{th} order of the given group is reduced to that of finding those of zero order in a group where the variables are $z, x_1, \ldots, x_n$, and the derivatives of z up to the k^{th} order.

§ 252. *Example.* As a very simple example, let it be required to find the differential invariants of the third order for the group

$$x' = x, \quad y' = \frac{ay+b}{cy+d}.$$

The linear operators of this group are

$$\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y}, \quad y^2\frac{\partial}{\partial y}.$$

Now $\eta\dfrac{\partial}{\partial y}$ extended to the third order is

$$\eta\frac{\partial}{\partial y} - \left(y_1\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial y_1} - \left(y_1^2\frac{\partial^2\eta}{\partial y^2} + y_2\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial y_2}$$
$$- \left(y_1^3\frac{\partial^3\eta}{\partial y^3} + 3y_1y_2\frac{\partial^2\eta}{\partial y^2} + y_3\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial y_3},$$

where we denote the first three derivatives of y with respect to x by y_1, y_2, y_3 respectively.

If we let η successively take the values $1, y, y^2$, we see that the functions we require must be annihilated by the three operators

$$\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y} - y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2} - y_3\frac{\partial}{\partial y_3},$$
$$y^2\frac{\partial}{\partial y} - 2y_1y\frac{\partial}{\partial y_1} - (2y_1^2 + 2yy_2)\frac{\partial}{\partial y_2} - (6y_1y_2 + 2y_3y)\frac{\partial}{\partial y_3},$$

and therefore also by the three unconnected operators

$$\frac{\partial}{\partial y}, \quad y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial y_3}, \quad y_1\frac{\partial}{\partial y_2} + 3y_2\frac{\partial}{\partial y_3}.$$

It follows that any function of x and $\dfrac{2y_1y_3 - 3y_2^2}{y_1^2}$ will be a differential invariant of the required class.

It may similarly be shown, by further extending the operators, that a differential invariant of the fourth order will

have the three annihilators

$$\frac{\partial}{\partial y}, \quad y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}+y_4\frac{\partial}{\partial y_4},$$
$$y_1^2\frac{\partial}{\partial y_2}+3y_1y_2\frac{\partial}{\partial y_3}+(4y_1y_3+3y_2^2)\frac{\partial}{\partial y_4};$$

that is, it will not involve y, will be homogeneous and of zero degree in y_1, y_2, y_3, y_4, and will be annihilated by the operator;

$$y_1^2\frac{\partial}{\partial y_2}+3\,y_1y_2\frac{\partial}{\partial y_3}+(4\,y_1y_3+3y_2^2)\frac{\partial}{\partial y_4}.$$

So also the invariant of the fifth order will not involve y, will be homogeneous, and of zero degree in $y_1, \ldots, y_5$, and will have the annihilator

$$y_1^2\frac{\partial}{\partial y_2}+3y_1y_2\frac{\partial}{\partial y_3}+(4y_1y_3+3y_2^2)\frac{\partial}{\partial y_4}+(5\,y_1y_4+10\,y_2y_3)\frac{\partial}{\partial y_5};$$

and so on, the new coefficient of the next highest partial operator being derived from the last by differentiating it totally with respect to x, and adding unity to the coefficient of y_1y_5 obtained by such differentiation.

§ 253. We shall now write down the extended operators of the projective group of the plane

(1) $\dfrac{\partial}{\partial x}$; (2) $\dfrac{\partial}{\partial y}$;

(3) $x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}-y_2\dfrac{\partial}{\partial y_2}-2\,y_3\dfrac{\partial}{\partial y_3}-3y_4\dfrac{\partial}{\partial y_4}-\ldots;$

(4) $x\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}-2\,y_1\dfrac{\partial}{\partial y_1}-3\,y_2\dfrac{\partial}{\partial y_2}-4\,y_3\dfrac{\partial}{\partial y_3}-\ldots;$

(5) $y\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial y}-(1+y_1^2)\dfrac{\partial}{\partial y_1}-3y_1y_2\dfrac{\partial}{\partial y_2}$

$$-(4\,y_1y_3+3\,y_2^2)\frac{\partial}{\partial y_3}-\ldots,$$

the coefficient of $-\dfrac{\partial}{\partial y_4}$ being obtained from that of $-\dfrac{\partial}{\partial y_3}$ by differentiating the latter totally with respect to x, and adding unity to the coefficient of y_1y_4 in the result, and so on;

(6) $y\dfrac{\partial}{\partial x}+x\dfrac{\partial}{\partial y}-(y_1^2-1)\dfrac{\partial}{\partial y_1}-3\,y_1y_2\dfrac{\partial}{\partial y_2}-\ldots,$

all terms after the third being the same as in (5);

$$(7)\quad x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}-(xy_1-y)\frac{\partial}{\partial y_1}-3\,xy_2\frac{\partial}{\partial y_2}$$
$$-(5xy_3+3\,y_2)\frac{\partial}{\partial y_3}-\ldots,$$

the coefficient of $-\dfrac{\partial}{\partial y_r}$ being formed by adding x^2y_r to the coefficient of $-\dfrac{\partial}{\partial y_{r-1}}$, differentiating the result totally with respect to x, and omitting the highest derivative in the result;

$$(8)\quad xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}-(xy_1{}^2-yy_1)\frac{\partial}{\partial y_1}-3xy_1y_2\frac{\partial}{\partial y_2}$$
$$-(4\,xy_1y_3+3xy_2{}^2+3y_1y_2+yy_3)\frac{\partial}{\partial y_3}-\ldots,$$

the coefficients of the successive terms being derived from the preceding ones as in (7), only that instead of adding x^2y_r we add xyy_r.

We could now find the invariant differential equations and the differential invariants up to any assigned order of this group, or of any of its sub-groups. Thus (1) and (2) form a sub-group of which any function of the derivatives not containing x or y is an invariant; (1), (2), (3), (4) form a sub-group of which any function of the derivatives $y_1, y_2, \ldots$ which is of zero degree and of zero weight will be an invariant; (1), (2), (5) is the group of movements in the plane with the geometrically obvious invariants $\rho, \dfrac{d\rho}{ds}, \dfrac{d^2\rho}{ds^2}, \ldots$, where ρ is the expression for the radius of curvature in Cartesian coordinates.

In order to obtain the differential invariants of a less obvious group we take (1), (2), (3), (4), and (7) which is at once seen to generate a sub-group. A differential invariant of this sub-group must be a function of $y_1, y_2, \ldots$ of zero degree and of zero weight; the only other condition which this function has to satisfy is that of being annihilated by

$$(9)\quad 3\,y_2\frac{\partial}{\partial y_3}+8\,y_3\frac{\partial}{\partial y_4}+15\,y_4\frac{\partial}{\partial y_5}+\ldots+r\,(r-2)\,y_{r-1}\frac{\partial}{\partial y_r}+\ldots.$$

It can be at once verified that the operator (9) annihilates I_4, I_5, I_6, I_7, where

$$I_4 = 3\,y_2y_4 - 4\,y_3^2,\quad I_5 = 9\,y_2^2y_5 + 40\,y_3^3 - 45\,y_2y_3y_4,$$
$$I_6 = 3\,y_2^3y_6 - 24\,y_2^2y_3y_5 + 60\,y_2y_3^2y_4 - 40\,y_3^4,$$
$$I_7 = 27\,y_2^4y_7 - 315\,y_2^3y_3y_6 + 1260\,y_2^2y_3^2y_5 - 2100\,y_2^2y_3^2y_4 + 1120\,y_3^5.$$

Each of the equations $I_4 = 0$, $I_5 = 0$, ... is invariant under the operations of this sub-group; and one of these, $I_5 = 0$, is invariant under all the operations of the general projective group of the plane. This last result is obvious from the geometrical fact that $I_5 = 0$ is the differential equation of the conic given by the general equation of the second degree in Cartesian coordinates. That $y_2 = 0$ is an invariant equation of the general projective group is also obvious geometrically.

The differential invariants of the sub-group (1), (2), (3), (4), (7), as distinguished from the invariant differential equations of the sub-group, are up to the 7th order

$$\frac{I_5^2}{I_4^3},\quad \frac{I_6}{I_4^2},\quad \frac{I_7^2}{I_4^5},\quad \frac{y_1^2 I_4}{y_2^4}.$$

What we have called invariant differential equations are sometimes called differential invariants; in such a notation our differential invariants are called *absolute differential invariants.*

§ 254. We now wish to find the differential invariant of lowest order of the general projective group of the plane.

We anticipate 'by counting the constants' that it will be of the 7th order; for there are eight operators in the group, and we do not therefore expect an invariant till these operators are extended so as to be in nine variables, and thus the derivatives of the 7th order will be involved. We shall find that this anticipation will be verified.

From (1) and (2) of § 253 we see that the invariant cannot contain x or y; and from (5) and (6) of the same article we know that it will not contain y_1; it must therefore be a function of

$$\frac{I_5^2}{I_4^3},\quad \frac{I_6}{I_4^2}\ \text{and}\ \frac{I_7^2}{I_4^5},$$

since an invariant of the group must clearly be an invariant of any sub-group, and therefore of the sub-group (1), (2), (3), (4), (7).

If we now extend all the operators to the 7th order we shall find that there are two additional operators to be added to

(3), (4), and (9) of § 253; and that the invariant, which is a function of $y_2, \ldots, y_7$ of zero degree and of zero weight, must be annihilated by these operators. These new operators are, omitting the parts of these operators which are connected with (3), (4), and (9), (we may do this since these parts will necessarily annihilate the invariant),

$$(10)\quad 6y_2\frac{\partial}{\partial y_4}+30y_2y_3\frac{\partial}{\partial y_5}+(60y_2y_4+40y_3^2)\frac{\partial}{\partial y_6}+(105y_2y_5+175y_3y_4)\frac{\partial}{\partial y_7},$$

and

$$(11)\quad 2y_2y_3\frac{\partial}{\partial y_4}+10y_3^2\frac{\partial}{\partial y_5}+(35y_3y_4-3y_2y_5)\frac{\partial}{\partial y_6}+(56y_3y_5+35y_4^2-7y_2y_6)\frac{\partial}{\partial y_7}.$$

The linear operator

$$(12)\qquad I_5\frac{\partial}{\partial y_6}+\frac{7}{3}\left(\frac{dI_5}{dx}\right)\frac{\partial}{\partial y_7}$$

(where $\left(\frac{dI_5}{dx}\right)$ denotes the total derivative of I_5 with respect to x) is connected with (10) and (11); and therefore we may replace the annihilator (11) of the required invariant by the annihilator (12).

Denoting the operators (10) and (12) respectively by X and Y the invariant required is a function of

$$\frac{I_5^2}{I_4^3},\quad \frac{I_6}{I_4^2},\quad \frac{I_7^2}{I_4^5}$$

annihilated by X and Y.

Now we easily verify that

$$XI_4=18y_2^3,\quad XI_5=0,\quad XI_6=60y_2^3I_4,\quad XI_7=315y_2^3I_5,$$

and therefore X annihilates P and Q, where

$$P=\frac{3I_6-5I_4^2}{I_5^{\frac{4}{3}}},\quad Q=\frac{2I_7-35I_4I_5}{I_5^{\frac{5}{3}}};$$

and the invariant required will be that function of P and Q which is annihilated by Y.

Now we may verify that

$$YP=\frac{9y_2^3}{I_5^{\frac{1}{3}}},\quad YQ=\frac{126y_2^4}{I_5^{\frac{5}{3}}}\left(\frac{dI_5}{dx}\right)-\frac{630}{I_5^{\frac{2}{3}}}y_2^3y_3;$$

and also that $\quad y_2\left(\frac{dI_5}{dx}\right)-5y_3I_5=3I_6-5I_4.$

We then have

$$YQ = \frac{126 y_2^3}{I_5^{\frac{5}{3}}} P \quad \text{and} \quad Y(7P^2 - Q) = 0;$$

and therefore $7P^2 - Q$ is the invariant which we require; that is,

$$(13) \qquad \frac{175 I_4^4 - 210 I_4^2 I_6 + 63 I_6^2 + 35 I_5^2 I_4 - 2 I_5 I_7}{I_5^{\frac{8}{3}}}$$

(where I_4, I_5, I_6, I_7 are as defined in § 253) is the differential invariant of lowest order for the general projective group of the plane.

From this invariant we can deduce the differential equation satisfied by all cuspidal cubics. To obtain this equation we reduce the cubic by a projective transformation to the form $y^2 = x^3$, and we therefore have

$$y = x^{\frac{3}{2}}, \quad y_1 = \tfrac{3}{2} x^{\frac{1}{2}}, \quad y_2 = \tfrac{3}{4} x^{-\frac{1}{2}}, \quad y_3 = -\tfrac{3}{8} x^{-\frac{3}{2}}.$$

If we now calculate for this cubic the values of I_4, I_5, I_6, I_7, and if we let I denote the numerator in (13), we have with little labour

$$2^{16}.\, 10^2.\, I^3 + 7^3.\, 3^6.\, I_5^{\,8} = 0;$$

and, as this equation is invariant for any projective transformation, it is zero for a cuspidal cubic, given by any equation in Cartesian coordinates.

§ 255. As an example in finding invariants of groups in three-dimensional space, we might take the group of movements of a rigid body, viz.

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}, \quad y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

and we should thus obtain the invariant differential equation of the first order

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 0;$$

and two differential invariants of the second order, viz. the expressions for the sum and product of the two principal radii of curvature at any point of a surface.

Since, however, these results are obvious geometrically we shall consider instead the invariants of the group

$$\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}, \quad x^2\frac{\partial}{\partial x} + (xy - z)\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\quad y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},\quad (xy-z)\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z};$$

these are the operators of the group of movements of a rigid body in non-Euclidean space.

Taking as usual p, q, r, s, t to denote the first and second derivatives of z with respect to x and y, the twice extended linear operator
$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}$$
is
$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z}+\pi\frac{\partial}{\partial p}+\kappa\frac{\partial}{\partial q}+\rho\frac{\partial}{\partial r}+\sigma\frac{\partial}{\partial s}+\tau\frac{\partial}{\partial t},$$
where (denoting by the suffixes 1, 2, 3 the partial derivatives of ξ, η, or ζ, with respect to x, y, z, respectively)

$$-\pi=p^2\xi_3+pq\eta_3+p(\xi_1-\zeta_3)+q\eta_1-\zeta_1,$$
$$-\kappa=q^2\eta_3+pq\xi_2+q(\eta_2-\zeta_3)+p\xi_2-\zeta_2,$$
$$\begin{aligned}-\rho=p^3\xi_{33}+p^2q\eta_{33}+p^2(2\xi_{13}-\zeta_{33})+2pq\eta_{13}+p(\xi_{11}-2\zeta_{13})+q\eta_{11}\\ -\zeta_{11}+2r(\xi_1+p\xi_3)+2s(\eta_1+p\eta_3)+r(p\xi_3+q\eta_3-\zeta_3),\end{aligned}$$
$$\begin{aligned}-\sigma=p^2q\xi_{33}+pq^2\eta_{33}+p^2\xi_{23}+q^2\eta_{13}+pq(\xi_{13}+\eta_{23}+\zeta_{33})\\ +p(\xi_{12}-\zeta_{23})+q(\eta_{12}-\zeta_{13})-\zeta_{12}\\ +s(\xi_1+\eta_2-\zeta_3+2p\xi_3+2q\eta_3)+r(\xi_2+q\xi_3)+t(\eta_1+p\eta_3),\end{aligned}$$
$$\begin{aligned}-\tau=q^3\eta_{33}+q^2p\xi_{33}+q^2(2\eta_{23}-\zeta_{33})+2pq\xi_{23}+q(\eta_{22}-2\zeta_{23})\\ +p\xi_{22}-\zeta_{22}+2t(\eta_2+q\eta_3)+2s(\xi_2+q\xi_3)+t(p\xi_3+q\eta_3-\zeta_3).\end{aligned}$$

There are six sets of values of ξ, η, ζ, viz.

(1)	$\xi=1$,	$\eta=0$,	$\zeta=y$,
(2)	$\xi=x$,	$\eta=0$,	$\zeta=z$,
(3)	$\xi=x^2$,	$\eta=xy-z$,	$\zeta=xz$,
(4)	$\xi=0$,	$\eta=1$,	$\zeta=z$,
(5)	$\xi=0$,	$\eta=y$,	$\zeta=z$,
(6)	$\xi=xy-z$,	$\eta=y_2$,	$\zeta=yz$.

Forming by aid of the above formulae the corresponding values of π, κ, ρ, σ, τ, we get the six operators

(1) $\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial z}+\dfrac{\partial}{\partial q}$;

(2) $x\dfrac{\partial}{\partial x}+z\dfrac{\partial}{\partial z}+q\dfrac{\partial}{\partial q}-r\dfrac{\partial}{\partial r}+t\dfrac{\partial}{\partial t}$;

(3) $$x^2\frac{\partial}{\partial x}+(xy-z)\frac{\partial}{\partial y}+xz\frac{\partial}{\partial z}-(px+qy-z-pq)\frac{\partial}{\partial p}+q^2\frac{\partial}{\partial q}-\big(r(3x-q)+2s(y-p)\big)\frac{\partial}{\partial r}-\big(2s(x-q)+t(y-p)\big)\frac{\partial}{\partial s}-t(x-3q)\frac{\partial}{\partial t};$$

(4) $$\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}+\frac{\partial}{\partial p};$$

(5) $$y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+p\frac{\partial}{\partial p}-t\frac{\partial}{\partial t}+r\frac{\partial}{\partial r};$$

(6) $$(xy-z)\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+yz\frac{\partial}{\partial z}-(px+qy-z-pq)\frac{\partial}{\partial q}+p^2\frac{\partial}{\partial p}-r(y-3p)\frac{\partial}{\partial r}-\big(2s(y-p)+r(x-q)\big)\frac{\partial}{\partial s}-\big(t(3y-p)+2s(x-q)\big)\frac{\partial}{\partial t}.$$

§ 256. As we have six operators forming a complete system in eight variables we expect two differential invariants of the second order; and could not have more, unless the six operators are connected; and it is easily seen that they are unconnected.

From (1) and (4) we see that the invariants must be functions of $p-y$, $q-x$, $z-xy$, r, s, and t; we therefore write

$$P=p-y,\quad Q=q-x,\quad Z=z-xy.$$

The operator (2) now takes the form

$$Q\frac{\partial}{\partial Q}+Z\frac{\partial}{\partial Z}-r\frac{\partial}{\partial r}+t\frac{\partial}{\partial t};$$

and (5) the form

$$P\frac{\partial}{\partial \rho}+Z\frac{\partial}{\partial Z}+r\frac{\partial}{\partial r}-t\frac{\partial}{\partial t};$$

while (3) becomes

$$(2Z+PQ)\frac{\partial}{\partial P}+Q^2\frac{\partial}{\partial Q}+(rQ+2sP)\frac{\partial}{\partial r}+(2sQ+tP)\frac{\partial}{\partial s}+3tQ\frac{\partial}{\partial t}+2x\left(Q\frac{\partial}{\partial Q}+Z\frac{\partial}{\partial Z}-r\frac{\partial}{\partial r}+t\frac{\partial}{\partial t}\right);$$

and we have a similar expression for (6).

It is now convenient to denote P by p, Q by q, and Z by z; in this notation we see that the invariants are functions of p, q, z, r, s, t, annihilated by each of the four operators

$$(\alpha)\quad q\frac{\partial}{\partial q}+z\frac{\partial}{\partial z}-r\frac{\partial}{\partial r}+t\frac{\partial}{\partial t},$$

$$(\beta)\quad p\frac{\partial}{\partial p}+z\frac{\partial}{\partial z}+r\frac{\partial}{\partial r}-t\frac{\partial}{\partial t},$$

$$(\gamma)\quad (2z+pq)\frac{\partial}{\partial p}+q^2\frac{\partial}{\partial q}+(rq+2sp)\frac{\partial}{\partial r}$$
$$+(2sq+tp)\frac{\partial}{\partial s}+3tq\frac{\partial}{\partial t},$$

$$(\delta)\quad (2z+pq)\frac{\partial}{\partial q}+p^2\frac{\partial}{\partial p}+(tp+2sq)\frac{\partial}{\partial t}$$
$$+(2sp+rq)\frac{\partial}{\partial s}+3rp\frac{\partial}{\partial r},$$

which we denote respectively by Ω_1, Ω_2, Ω_3, and Ω_4.

We have

$$\Omega_1(pq+z)=pq+z,\quad \Omega_2(pq+z)=pq+z,$$
$$\Omega_3(pq+z)=2q(z+pq),\quad \Omega_4(pq+z)=2p(z+pq),$$

so that the equation $z+pq=0$ is invariant (or in the original notation $z+pq=px+qy$).

Also
$$\Omega_1 rq^2=rq^2,\quad \Omega_2 rq^2=rq^2,$$
$$\Omega_3 rq^2=q^2(3rq+2sp),\quad \Omega_4 rq^2=(4z+5pq)rq;$$

and forming similar equations for tp^2 and $s(pq+2z)$ we see that

$$\Omega_1\left(rq^2+tp^2-2s(pq+2z)\right)=rq^2+tp^2-2s(pq+2z);$$
$$\Omega_2\left(rq^2+tp^2-2s(pq+2z)\right)=rq^2+tp^2-2s(pq+2z);$$
$$\Omega_3\left(rq^2+tp^2-2s(pq+2z)\right)=3q\left(rq^2+tp^2-2s(pq+2z)\right);$$
$$\Omega_4\left(rq^2+tp^2-2s(pq+2z)\right)=3p\left(rq^2+tp^2-2s(pq+2z)\right).$$

Since
$$\Omega_1(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}}=(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}}=\Omega_2(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}},$$
$$\Omega_3(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}}=3q(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}},$$
$$\Omega_4(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}}=3p(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}},$$

we can therefore see that

$$\frac{rq^2+tp^2-2s(pq+2z)}{(pq+z)^{\frac{3}{2}}z^{-\frac{1}{2}}}$$

is a differential invariant for the group.

It may be similarly proved that

$$\frac{rt-s^2}{(pq+z)^2 z^{-2}}$$

is the other differential invariant of the group.

In the original notation, therefore, the invariants are

$$\frac{r(q-x)^2+t(p-y)^2-2s(2z+pq-px-qy-xy)}{(z+pq-px-qy)^{\frac{3}{2}}(z-xy)^{-\frac{1}{2}}},$$

and

$$\frac{rt-s^2}{(z+pq-px-qy)^2(z-xy)^{-2}}.$$

§ 257. These examples indicate that the only difficulty in obtaining differential invariants of a given group is the difficulty of finding the solutions of a given complete system of equations.

We are often much helped by geometrical considerations; thus in the example just considered we knew that the group was a projective one in ordinary three-dimensional space; and we knew that it transformed the quadric $z = xy$ into itself. If then from any point P on a surface S we draw the tangent cone to this quadric it will meet the tangent plane at P to the surface S in a pair of lines; these lines, together with the inflexional tangents to S at P, will form a pencil of four rays. The condition that the pencil should be harmonic is unaltered by any projective transformation, and is, in the notation here employed,

$$r(q-x)^2+t(p-y)^2-2s(2z+pq-px-qy-xy) = 0.$$

Similarly the condition that the surface S should be a developable is unaltered by projective transformation, and is $rt-s^2 = 0$.

It was by attending to these considerations that one was enabled to simplify the solution of the given complete system.

CHAPTER XXI

THE GROUPS OF THE STRAIGHT LINE, AND THE PRIMITIVE GROUPS OF THE PLANE

§ 258. When we are given the structure constants of a group we have seen how the types of groups with the required structure are to be formed. If, instead of being given the structure constants, we are merely given the order r of the group required, we should have to find the sets of r^3 constants which will satisfy the equations

$$c_{ijk} + c_{jik} = 0,$$

$$\sum_{h=r} (c_{ikh}\, c_{jhm} + c_{kjh}\, c_{ihm} + c_{jih}\, c_{khm}) = 0,$$

where the suffixes i, k, j, m may have any values from 1 to r.

Two sets of constants $c'_{ijk}, \ldots$ and $c_{ijk}, \ldots$ satisfying these equations would not be considered distinct structure sets if they could be connected by the equation system

$$\sum^{h=r} a_{hs}\, c'_{ikh} = \sum^{p=q=r} a_{ip}\, a_{kj}\, c_{pqs},$$

where $a_{ij}, \ldots$ is a set of constants whose determinant

$$\begin{vmatrix} a_{11}, & \cdot & \cdot & \cdot & a_{1r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{r1}, & \cdot & \cdot & \cdot & a_{rr} \end{vmatrix}$$

does not vanish, as we explained in Chapter V.

Suppose however that, instead of being given the order of the group, we are given the number of variables in the operators of the groups, how are we to find all possible types of groups in these variables? The method of finding the structure constants is not now available; for, when the number of variables, n, is greater than unity, the order of the group, r,

may have any value up to infinity. The problem suggested has so far only been solved for the cases $n = 1$, $n = 2$, $n = 3$. In this chapter it will be shown how the groups of the straight line, and the primitive groups of the plane may be obtained.

§ 259. A group $X_1, \ldots, X_r$, where

$$X_k = \xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, r),$$

is transitive if it has n unconnected operators; that is, if not all n-rowed determinants vanish identically in the matrix

$$\left\| \begin{matrix} \xi_{11}, & \cdot & \cdot & \cdot & \xi_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_{r1}, & \cdot & \cdot & \cdot & \xi_{rn} \end{matrix} \right\|.$$

Now let $x_1^0, \ldots, x_n^0$ be a point of general position, that is, a point whose coordinates do not make all n-rowed determinants vanish in the matrix, and in the neighbourhood of which all the functions $\xi_{ij}, \ldots$ are holomorphic. By transforming to parallel axes through this point we may expand all the functions $\xi_{ij}, \ldots$, in powers of $x_1, \ldots, x_n$; and we then see that from the r operators of the group a set of n independent ones, say $X_1, \ldots, X_n$, can be selected such that

$$X_k = \frac{\partial}{\partial x_k} + \xi_{k1}\frac{\partial}{\partial x_1} + \ldots + \xi_{kn}\frac{\partial}{\partial x_n}, \qquad (k = 1, \ldots, n),$$

where ξ_{ki} vanishes for $x_1 = 0, \ldots, x_n = 0$.

The other $(r-n)$ operators of the group $X_{n+1}, \ldots, X_r$ may be so chosen that for each of them ξ_{ij}, when expanded, has no term not beginning with powers of $x_1, \ldots, x_n$, that is, no constant term. These $(r-n)$ operators form a sub-group, the *group of the origin*, characterized by the property of leaving the origin at rest.

If in an operator

$$\xi_1\frac{\partial}{\partial x_1} + \ldots + \xi_n\frac{\partial}{\partial x_n}$$

the lowest powers of $x_1, \ldots, x_n$ which occur when $\xi_1, \ldots, \xi_n$ are expanded are of degree s, then we say that *the operator is of degree s*.

If we have a number of operators $Y_1, \ldots, Y_q$ each of degree s, and if no operator *dependent* on these, that is, of the form

$$e_1 Y_1 + \ldots + e_q Y_q,$$

where $e_1, \ldots, e_q$ are constants, is of higher degree than s, we say that they form a system of degree s. It is clear that we cannot have more than n operators in a system of degree zero nor more than n^2 in one of degree unity, and so on.

If then the operators $X_{n+1}, \ldots, X_r$ do not form a system of degree unity, we can deduce from them a number of operators of the second degree; and proceeding similarly with these latter we may be able to deduce a system of the third degree, and so on.

We therefore see that the operators of a transitive group may be arranged as follows: n operators forming a system of zero degree, m_1 forming a system of the first degree, m_2 a system of the second degree, ..., m_s a system of s^{th} degree.

Since all of these operators are independent, and the group is finite, s cannot exceed a finite limit, and we have

$$r = n + m_1 + \ldots + m_s.$$

If we form the alternant of two operators of degrees p and q respectively, it can be at once verified that it cannot be of degree lower than $p+q-1$. This principle is of great use in determining the possible types of groups when n is fixed; we shall now apply it to obtain the possible finite continuous groups in a single variable, that is, the groups of the straight line.

First, we notice that if a group contains no operator of degree k, then it cannot contain one of degree $(k+1)$; for it must have, if transitive, n operators of zero degree, and, by forming the alternants of these with the operators of degree $(k+1)$, we must have operators of degree k.

§ 260. We now consider the case where n is unity; we may take the operators of such a group to be

$$\frac{\partial}{\partial x} + \xi_1 \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial x}, \ldots, x^s \frac{\partial}{\partial x} + \xi_{s+1} \frac{\partial}{\partial x},$$

where ξ_i contains x in degree i at the lowest; and in this group there must be no operator of degree higher than s.

Suppose that $s > 2$; then, forming the alternant of the operators of degree s and $(s-1)$ respectively, the group must contain an operator of degree $(2s-2)$, viz.

$$x^{2s-2} \frac{\partial}{\partial x} + \xi_{2s-1} \frac{\partial}{\partial x},$$

which, since $s > 2$, would be an operator of degree higher than s;

and, as this is impossible, we conclude that s cannot be greater than two.

A group in a single variable cannot then contain more than three independent operators.

A general principle, whatever may be the number of variables, is that all operators of the k^{th} and higher degrees form a sub-group. This is proved from the fact that any two such operators have an alternant whose degree is at least $(2k-1)$, and therefore not less than k, unless k is zero; if k is zero the operators of the k^{th} and higher degrees form the group itself.

If from the operators $X_1, \ldots, X_r$ we form a new set of operators, by adding to any operator of degree k any operator dependent on the operators of degree not less than k, we shall still have the operators of the group arranged in systems of degree zero to s. Advantage of this principle may often be taken to simplify the structure constants of a group.

Thus in the case of a single variable, suppose $s = 2$, and let X_0, X_1, X_2 be the three independent operators respectively of degrees 0, 1, 2. From the group property we have

$$(X_1, X_2) = aX_0 + bX_1 + cX_2,$$

where a, b, c are constants.

Since (X_1, X_2) is of the second degree, a and b must be zero; and, by comparing the coefficients of $\frac{\partial}{\partial x}$ on the two sides of the identity, we see that c is unity.

Similarly we see that

$$(X_0, X_2) = 2X_1 + eX_2,$$

where e is some unknown constant.

To eliminate this constant, we take as the operators of the group Y_0, Y_1, Y_2 where

$$Y_0 = X_0, \quad Y_1 = X_1 + \tfrac{1}{2}eX_2, \quad Y_2 = X_2,$$

and we have

$$(1) \qquad (Y_1, Y_2) = Y_2, \quad (Y_0, Y_2) = 2Y_1.$$

Suppose now that

$$(2) \qquad (Y_0, Y_1) = Y_0 + aY_1 + bY_2,$$

where a and b are some unknown constants: from Jacobi's identity

$$((Y_0, Y_1), Y_2) + ((Y_1, Y_2), Y_0) + ((Y_2, Y_0), Y_1) = 0,$$

and therefore from (1) and (2)

$$a(Y_1, Y_2) = 0,$$

so that a is zero.

We now take (β being an undetermined constant)

$$Z_0 = Y_0 + \beta Y_2, \quad Z_1 = Y_1, \quad Z_2 = Y_2,$$

and have

$$(Z_0, Z_2) = 2Z_1, \quad (Z_1, Z_2) = Z_2, \quad (Z_0, Z_1) = Z_0 + (b - 2\beta) Z_2;$$

and therefore, by taking $2\beta = b$, we see that the group has three operators Z_0, Z_1, Z_2 respectively of degrees 0, 1, 2, and of the structure

$$(Z_0, Z_1) = Z_0, \quad (Z_1, Z_2) = Z_2, \quad (Z_0, Z_2) = 2Z_1.$$

By a change of the variable from x to x' we can reduce $\frac{\partial}{\partial x} + \xi_1 \frac{\partial}{\partial x}$ to the form $\frac{\partial}{\partial x'}$; to do this we have $\frac{dx'}{dx} = \frac{1}{1+\xi_1}$, where ξ_1 is of degree unity in x at least, and we may take x' in the form $x + f(x)$, where $f(x)$ is a holomorphic function of x, whose lowest term is of the second degree in x at least. In the new variables therefore Z_0, Z_1, Z_2 will still be of degrees 0, 1, 2 respectively, but ξ_1 will be identically zero.

Omitting accents from the variable we take Z_0 to be $\frac{\partial}{\partial x}$.

Since

$$\left(\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x},$$

we see that ξ_2 must be a mere constant; it must therefore be zero, since it was given to be at least of the second degree in x. We may similarly deduce that ξ_3 is zero; and therefore the only group of the third order is

$$\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x}.$$

Similarly we may see that the only group of order 2 is of the type

$$\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x};$$

and the only group of order unity is $\frac{\partial}{\partial x}$.

§ 261. Before applying this method to find the types of groups in two variables, it will be convenient to consider how,

by a linear transformation of the variables, the operator

$$(1)\quad (a_{11}x_1+\dots+a_{1n}x_n)\frac{\partial}{\partial x_1}+\dots+(a_{n1}x_1+\dots+a_{nn}x_n)\frac{\partial}{\partial x_n}$$

may be reduced to a simple form.

Let λ_1 be any root of the equation

$$\begin{vmatrix} a_{11}-\lambda, & a_{21}, & \cdot & \cdot & a_{n1} \\ a_{12}, & a_{22}-\lambda, & \cdot & \cdot & a_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1n}, & a_{2n}, & \cdot & \cdot & a_{nn}-\lambda \end{vmatrix} = 0;$$

and let us find n quantities $e_1, \dots, e_n$ such that

$$\begin{aligned} a_{11}e_1+\dots+a_{n1}e_n &= \lambda_1 e_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}e_1+\dots+a_{nn}e_n &= \lambda_1 e_n. \end{aligned}$$

These quantities will, unless all first minors of the determinant vanish, be proportional to the first minors of any row.

We take as a variable to replace some one of the set $x_1, \dots, x_n$, say x_1, the expression y_1 where

$$y_1 = e_1x_1+\dots+e_nx_n.$$

We then see that the operator (1) is of the same form in the variables $y_1, x_2, \dots, x_n$ as it was in $x_1, \dots, x_n$, but the constants $a_{ij}, \dots$ are replaced by a new set of constants $a'_{ij}, \dots$ characterized by the property

$$a'_{11} = \lambda_1,\ a'_{12} = 0, \dots, a'_{1n} = 0.$$

By a linear transformation, then, the operator (1) can be reduced to such a form that

$$a_{11} = \lambda_1,\ a_{12} = 0, \dots, a_{1n} = 0.$$

We similarly see that, by introducing a new variable y_2 where

$$y_2 = e_2x_2+\dots+e_nx_n,$$

and $e_2, \dots, e_n$ are determined by

$$\begin{aligned} a_{22}e_2+\dots+a_{n2}e_n &= \lambda_2 e_n, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \ \cdot\ \cdot\ \cdot\ \cdot \\ a_{2n}e_2+\dots+a_{nn}e_n &= \lambda_2 e_n, \end{aligned}$$

the operator can be still further reduced to a form in which, in addition to the former simplification, we have

$$a_{22} = \lambda_2,\ a_{23} = 0, \dots, a_{2n} = 0.$$

Proceeding thus we see that the operator can by linear transformation be reduced to the form

$$(2)\quad \lambda_1 x_1 \frac{\partial}{\partial x_1} + (a_{21}x_1 + \lambda_2 x_2)\frac{\partial}{\partial x_2} + (a_{31}x_1 + a_{32}x_2 + \lambda_3 x_3)\frac{\partial}{\partial x_3} + \dots.$$

This operator may be still further simplified; suppose λ_1 and λ_2 are unequal, and apply the transformation

$$y_1 = x_1,\ y_2 = x_2 + \lambda x_1,\ y_3 = x_3, \dots, y_n = x_n$$

which gives

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1} + \lambda \frac{\partial}{\partial y_2},\quad \frac{\partial}{\partial x_2} = \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n},$$

we then see that by a suitable choice of λ, without otherwise altering the form of (2), we can make the new a_{21} to be zero, when we express the operator (2) in terms of the new variables.

Similarly, having caused a_{21} to disappear, by a transformation of the form

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3 + \lambda x_1,\ y_4 = x_4, \dots, y_n = x_n,$$

we could cause a_{31} also to disappear from the new form of the operator; and proceeding thus, so long as none of the coefficients $\lambda_2, \dots, \lambda_n$ are equal to λ_1, we could cause $a_{41}, \dots, a_{n1}$ to disappear.

In exactly the same manner, by properly choosing the transformations, we could cause all the coefficients $a_{ij}, \dots$ to disappear so long as none of the quantities $\lambda_1, \dots, \lambda_n$ are equal; that is, if the determinant has no equal roots, the canonical form of the linear operator is

$$\lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2} + \dots + \lambda_n x_n \frac{\partial}{\partial x_n}.$$

§ 262. The general method of obtaining a canonical form for the case of equal roots will be sufficiently explained by considering the case where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, and no other root is equal to λ_1.

First consider the coefficient of $\frac{\partial}{\partial x_5}$; by the transformation

$$y_5 = x_5 + \lambda x_4,\ y_1 = x_1, \dots, y_n = x_n$$

we can by a suitable choice of λ cause a_{54} to disappear; and by a similar transformation we can cause a_{53}, a_{52}, a_{51} also to disappear.

It is thus seen that the operator may by a linear transformation be brought to such a form that x_1, x_2, x_3, x_4 only appear in the first four terms.

These terms take the form

$$\lambda_1\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4}\right)+x_1\left(a_{21}\frac{\partial}{\partial x_2}+a_{31}\frac{\partial}{\partial x_3}+a_{41}\frac{\partial}{\partial x_4}\right)$$
$$+a_{32}x_2\frac{\partial}{\partial x_3}+(a_{42}x_2+a_{43}x_3)\frac{\partial}{\partial x_4}.$$

Now by any linear transformation in x_1, x_2, x_3, x_4 the part

$$x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4}$$

is unaltered; if a_{21} is not zero by a transformation of the form

$$y_1=x_1,\quad y_2=x_2,\quad y_3=x_3,\quad y_4=x_4+\lambda x_2$$

we can eliminate the new a_{41}; we may then by a transformation

$$y_1=x_1,\quad y_2=x_2,\quad y_3=x_3+\lambda x_2,\quad y_4=x_4$$

eliminate a_{31}; and then, if a_{32} is not zero, we may eliminate a_{42}; while if a_{32} is zero by a transformation

$$y_1=x_1,\quad y_2=x_2+\lambda x_3,\quad y_3=x_3,\quad y_4=x_4$$

we may eliminate a_{43}.

If a_{21} is zero, but not a_{32}, we take

$$y_1=x_1,\quad y_2=a_{32}x_2+a_{31}x_1,\quad y_3=x_3,\quad y_4=x_4,$$

and thus eliminate a_{31}; if a_{21} and a_{32} are both zero, but not a_{43}, we take

$$y_1=x_1,\quad y_2=x_2,\quad y_3=a_{41}x_1+a_{42}x_2+a_{43}x_3,\quad y_4=x_4,$$

and thus eliminate a_{41} and a_{42}. Finally if a_{21}, a_{32}, and a_{43} are all zero, we can similarly eliminate a_{41}. Summing up we see that the first four terms may be reduced to the form

$$\lambda_1\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4}\right)$$
$$+e_1x_1\frac{\partial}{\partial x_2}+e_2x_2\frac{\partial}{\partial x_3}+e_3x_3\frac{\partial}{\partial x_4},$$

where e_1, e_2, e_3 are symbols for constants; and it is easily seen that, by further simple transformations, we may reduce

these constants to such forms that any one, which is not zero, is unity.

Similar expressions could be obtained for the other parts of the operator; and we thus see how, in any given number of variables, to write down all possible types of such operators.

We know of course that any linear operator can be reduced to the type $\frac{\partial}{\partial x}$; but such reduction is not effected by a *linear* transformation, and just now we are only considering how to obtain types by linear transformation; that is, types *conjugate within the general linear homogeneous group.*

§ 263. We now enumerate the types of linear homogeneous groups of order one in two variables x, y; we write p for $\frac{\partial}{\partial x}$ and q for $\frac{\partial}{\partial y}$, and e for an arbitrary constant:

(1) $e(xp+yq)+xp-yq$, (2) $xp+yq+xq$,

(3) $xp+yq$, (4) $xp-yq$, (5) xq.

We shall now find all possible types of linear groups of the third order.

First we find all the groups containing the operator (3) $xp+yq$; by a linear transformation every operator of the group we seek can be reduced to one of the above five forms (though the same transformation will not necessarily bring two operators of the group simultaneously to these normal forms); and a linear transformation cannot alter the form of (3).

Since we only require two operators to complete the group of the third order which contains (3); and, since these must be independent of (3), one of the operators may be taken to be of the form (4) or (5).

Suppose it is of the form (4), the remaining operator of the group must be of the form

$$a(xp+yq)+b(xp-yq)+cxq+dyp,$$

where a, b, c, d are constants; as we only require the part independent of (3) and (4), we may take a and b to be zero.

Form the alternant of (4) with

$$cxq+dyp,$$

and we shall see that

$$cxq-dyp$$

is an operator within the group. As the group is to be of the

third order, and to contain (3) and (4); and, as we now see that cxq and dyp are operators of the group, we must have, either d zero when the group is

$$(6) \qquad xp - yq, \quad xp + yq, \quad xq;$$

or c zero, when we get a group of the same type; that is, a group transformable into (6) by a linear transformation.

If we had assumed that the second operator was of the form (5) we should have been led to the same group (6).

We must now find the linear groups of the third order which do not contain the operator (3).

Suppose that one operator of our group is of the type (5); and let a second operator be

$$a(xp + yq) + b(xp - yq) + cyp.$$

Forming the alternant with xq we see that the group will contain

$$c(xp - yq);$$

first we suppose that c is zero; and we take the third operator of the group to be

$$(7) \qquad a_1(xp + yq) + b_1(xp - yq) + c_1 yp,$$

where a_1, b_1, c_1 are constants.

Now c_1 cannot be zero, for, if it were,

$$a(xp + yq) + b(xp - yq) \quad \text{and} \quad a_1(xp + yq) + b_1(xp - yq)$$

would be two independent operators of the group; and therefore $xp + yq$ would be an operator of the group, which is contrary to our hypothesis.

Forming the alternant of (7) and (5) we see that the group will contain

$$c_1(xp - yq),$$

and therefore the group which contains (5), and does not contain (3), must contain (4).

We therefore take the third operator of this group to be

$$a(xp + yq) + byp;$$

and forming the alternant with (4) we see that the group must contain yp, and we thus have the group

$$(8) \qquad xq, \ yp, \ xp - yq.$$

We obtain the same group by supposing the first operator to be of the type (4).

We have now only to find any possible group of the third order which does not contain any operator of the types (3), (4), or (5).

Suppose that one operator is of the type (2); we then take a second to be

$$a\,(xp+yq)+b\,(xp-yq)+cyp,$$

and the third

$$a_1\,(xp+yq)+b_1\,(xp-yq)+c_1\,yp,$$

and we may clearly suppose that either c or c_1 is zero; say we take c to be zero, if we now form the alternant of

$$a\,(xp+yq)+b\,(xp-yq)$$

with (2), we shall get an operator of the type (5), which is contrary to our hypothesis.

The group cannot therefore contain an operator of the type (2); and we see similarly that it cannot contain one of the type (1).

The only groups of the third order are therefore

$$xq,\ xp-yq,\ xp+yq,$$

and

$$xq,\ xp-yq,\ yp.$$

It may be shown in a similar manner that the only groups of the second order are

$$e\,(xp+yq)+xp-yq,\ \ xq\,;$$
$$xp-yq,\ \ xp+yq\,;$$
$$xq,\qquad xp+yq.$$

We have now found all possible sub-groups of the general linear group in x, y; we might have obtained these directly by the method explained in Chapter XIII.

§ 264. It is now necessary to examine the groups which we have found; and to see, with respect to each of them, whether there is any linear equation

$$\lambda x+\mu y=0$$

admitting all the transformations of the group.

It may be at once verified that the group

$$xq,\ xp-yq,\ xp+yq$$

is admitted by the equation $x=0$; that is, by any transformation of this group, points on the line $x=0$ are transformed so as still to remain on the line $x=0$.

It may similarly be proved by successively examining these groups that, for each group, at least one linear equation can be found to admit the transformations of that group, unless the group is either

(1) *the general linear group,*

$$xq,\ yp,\ xp-yq,\ xp+yq,$$

or (2) *the special linear group,*

$$xq,\ yp,\ xp-yq.$$

§ 265. We now proceed to determine the types of primitive groups of the plane.

If a group is imprimitive it must have at least one invariant equation of the form

$$\frac{dy}{dx}=\phi(x, y).$$

We express this condition geometrically by saying that an infinity of curves can be drawn on the plane; and that by the operations of the imprimitive group these curves are only interchanged *inter se*; any set of points, lying on one of the curves of the system, being transformed so as to be a set, lying on some other curve of the system.

If then we take a point of general position the *group of the point,* that is, the transformations of the imprimitive group which keep that point at rest, cannot alter the curve of the system which passes through the point; and in particular the direction of the curve at the point is not altered.

We take the origin to be a point of general position; then the lowest terms in the group of the origin are of the first degree; suppose P is the origin, and PT the tangent to any curve which passes through P; by the operations of the group of the origin this curve will be transformed into a system of curves all passing through P; and the directions of the tangents at P to these curves are what the direction PT has been transformed into by the operations of the group of the origin.

Now the only terms in the group which are effective in this transformation of the linear elements through P are the lowest terms; that is, the linear elements at P are transformed by a linear group.

We obtain this same result analytically as follows:—

let $$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}$$

be any operator of the group of the origin, so that ξ and η, the terms of lowest degree in x, y, are at least of the first

degree; and let us extend the operator (denoting by p the quantity $\frac{dy}{dx}$) so as to get

$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\left(\eta_1+p(\eta_2-\xi_1)-p^2\xi_2\right)\frac{\partial}{\partial p},$$

where the suffix 1 denotes partial differentiation with respect to x, and the suffix 2 partial differentiation with respect to y.

We are only concerned to know how the p of any line through the origin is transformed; this we know through the operator

$$\left(\eta_1+p(\eta_2-\xi_1)-p^2\xi_2\right)\frac{\partial}{\partial p},$$

where after the partial differentiations have been carried out we are to take $x=0$, $y=0$; we therefore need only consider those parts of ξ and η which are linear in x, y.

Now if the group is imprimitive at least one value of p can be found which is invariant for the group of the origin; but if the group is primitive no such value can be found. If therefore the group is primitive the operators in it of the first degree, according to the classification explained in § 259, must either be of the form

$$(1)\qquad y\frac{\partial}{\partial x}+\ldots,\quad x\frac{\partial}{\partial y}+\ldots,\quad x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+\ldots,$$

where the terms not written down but indicated by $+\ldots$ are of higher degree in the variables than those which are written down; or else they must be of the form

$$(2)\quad y\frac{\partial}{\partial x}+\ldots,\ x\frac{\partial}{\partial y}+\ldots,\ x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+\ldots,\ x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots;$$

for, by § 264, all other forms for the group of the origin would leave invariant at least one linear element through the origin.

§ 266. Suppose that the operators of the first degree are of the form (1); it will now be proved that there cannot be any operator of degree three, and therefore not any of higher degree.

Suppose that there could exist in the group the operator

$$(1)\qquad y^3\frac{\partial}{\partial x}+\ldots,$$

where the terms not written down are of higher degree than those written down; form its alternant with

$$x\frac{\partial}{\partial y}+\ldots,$$

when we shall see that the group must contain

(2) $$3xy^2\frac{\partial}{\partial x}-y^3\frac{\partial}{\partial y}+\ldots.$$

Forming the alternant of (1) and (2) we get

(3) $$y^5\frac{\partial}{\partial x}+\ldots,$$

and forming the alternant of (2) and (3) we get

$$y^7\frac{\partial}{\partial x}+\ldots;$$

and so on *ad infinitum*; so that the group would not be finite as all of these operators are independent.

We can now prove that there can be no operator

(4) $$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial x}+\ldots,$$

where ξ and η are of the third degree; forming the alternant of (4) with $y\frac{\partial}{\partial x}+\ldots$ we get

$$(y\xi_1-\eta)\frac{\partial}{\partial x}+y\eta_1\frac{\partial}{\partial y}+\ldots.$$

Forming the alternant of this again with $y\frac{\partial}{\partial x}+\ldots$, and so on, we get successively

$$(y^2\xi_{11}-2y\eta_1)\frac{\partial}{\partial x}+y\eta_{11}\frac{\partial}{\partial y}+\ldots,$$

$$(y^3\xi_{111}-3y^2\eta_{11})\frac{\partial}{\partial x}+y\eta_{111}\frac{\partial}{\partial y}+\ldots,$$

$$-4y^3\eta_{111}\frac{\partial}{\partial x}+\ldots.$$

Now η_{111} is a constant, and it must be zero, else would the group have an operator

$$y^3\frac{\partial}{\partial x}+\ldots,$$

and therefore η must contain y as a factor; similarly we see that ξ must contain x as a factor.

We must now try whether there can be an operator of the form

$$(5) \qquad x\xi\frac{\partial}{\partial x}+y\eta\frac{\partial}{\partial y}+\dots$$

where ξ and η are of the second degree; forming the alternant with $y\frac{\partial}{\partial x}+\dots$ we have

$$(6) \qquad \left(yx\xi_1+y(\xi-\eta)\right)\frac{\partial}{\partial x}+y^2\eta_1\frac{\partial}{\partial y}+\dots.$$

Now the coefficient of $\frac{\partial}{\partial x}$, being of the third degree, must be divisible by x; and therefore $\xi-\eta$ must be divisible by x; by symmetry it must be divisible by y, so that

$$\xi-\eta = axy,$$

where a is a constant.

The result at which we have arrived is that in any operator of the third degree

$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\dots,$$

$\xi\div x-\eta\div y$ is divisible by xy. Applying this theorem to (6), and writing $\eta+axy$ for ξ, we see that a is zero, so that ξ and η are equal.

We then have to try whether the group can contain an operator of the form

$$\xi\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\dots.$$

where ξ is of the second degree.

Forming its alternants with the operators of zero degree viz. $\frac{\partial}{\partial x}+\dots$, and $\frac{\partial}{\partial y}+\dots$, we obtain the two operators

$$\xi_1\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\xi\frac{\partial}{\partial x}+\dots,$$

$$\xi_2\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\xi\frac{\partial}{\partial y}+\dots;$$

and forming the alternant of these two we have

$$\xi\xi_2\frac{\partial}{\partial x}-\xi\xi_1\frac{\partial}{\partial y}+\dots.$$

This operator being of the third degree, must be such that

$$\frac{\xi_2}{x} = \frac{-\xi_1}{y};$$

and, ξ being of the second degree, we must therefore have

$$\xi_2 = kx, \quad \xi_1 = -ky,$$

where k is a constant.

Now
$$\frac{\partial}{\partial y}\xi_1 \equiv \frac{\partial}{\partial x}\xi_2$$

and therefore k must be zero; so that ξ being of the second degree and ξ_1 and ξ_2 both zero, ξ must vanish identically. We have therefore proved the theorem we enunciated, viz. that no operator of degree three can exist in the group.

§ 267. We have now to find the possible forms of operators of the second degree; let such an operator be

$$(1) \qquad \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \ldots.$$

First we could prove as before that the hypothesis of an operator of the form

$$y^2\frac{\partial}{\partial x} + \ldots$$

existing in the group would involve the non-finiteness of the group.

Form successive alternants of (1) with $y\frac{\partial}{\partial x} + \ldots$; and we get

$$y^2\eta_{11}\frac{\partial}{\partial x} + \ldots;$$

and therefore, since we must have η_{11} zero, we see that η contains y as a factor. Similarly we see that ξ contains x as a factor; and we need only consider operators of the form

$$(2) \qquad x\xi\frac{\partial}{\partial x} + y\eta\frac{\partial}{\partial y} + \ldots,$$

where ξ and η are of the first degree.

Form the alternant of (2) with $y\frac{\partial}{\partial x} + \ldots$, and we shall see that $\xi - \eta$ is divisible by x, and therefore by symmetry it is also divisible by y; but $\xi - \eta$ is of the first degree, and therefore must vanish identically.

The only possible operators of the second degree are therefore

$$\xi\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right),$$

where ξ is of degree unity.

So far the reasoning has only involved the existence of two of the operators of the first degree, viz.

$$(7)\qquad x\frac{\partial}{\partial y}+\ldots \text{ and } y\frac{\partial}{\partial x}+\ldots,$$

and it therefore applies equally to either class (1) or class (2) of the primitive groups.

We now assume that the group is of the first class and so has no operator of the form

$$(8)\qquad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots,$$

and we shall see that ξ must be zero.

Forming the alternants of

$$\xi\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots$$

with

$$\frac{\partial}{\partial x}+\ldots, \text{ and } \frac{\partial}{\partial y}+\ldots,$$

we have in the group the operators

$$(9)\qquad \begin{aligned}&\xi_1\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\xi\frac{\partial}{\partial x}+\ldots,\\ &\xi_2\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\xi\frac{\partial}{\partial y}+\ldots.\end{aligned}$$

Since ξ is linear and equal, say, to $ax+by$, the existence of (9) and (7) involves the existence of (8), unless a and b are zero.

A primitive group of the first class can then only have the five operators

$$\frac{\partial}{\partial x}+\ldots,\quad \frac{\partial}{\partial y}+\ldots,\quad x\frac{\partial}{\partial y}+\ldots,\quad y\frac{\partial}{\partial x}+\ldots,\quad x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+\ldots.$$

§ 268. We shall now for brevity denote by P the operator $y\frac{\partial}{\partial x}+\ldots$, by Q the operator $x\frac{\partial}{\partial y}+\ldots$, and by R the operator

$$x\frac{\partial}{\partial y}-y\frac{\partial}{\partial y}+\ldots.$$

P, Q, R is the group of the origin, and we have

$$(P, R) = 2P, \quad (Q, R) = -2Q, \quad (P, Q) = -R.$$

Also, since P, Q, R form with $\frac{\partial}{\partial x} + \ldots$ and $\frac{\partial}{\partial y} + \ldots$ the group itself,

$$\left(\frac{\partial}{\partial y} + \ldots, P\right) = a_1 P + b_1 Q + c_1 R + \frac{\partial}{\partial x} + \ldots,$$

$$\left(\frac{\partial}{\partial x} + \ldots, Q\right) = a_2 P + b_2 Q + c_2 R + \frac{\partial}{\partial y} + \ldots,$$

where a_1, b_1, c_1, a_2, b_2, c_2 are unknown constants.

If we now take as two operators of the group X and Y where

$$X = \alpha_1 P + \beta_1 Q + \gamma_1 R + \frac{\partial}{\partial x} + \ldots,$$

$$Y = \alpha_2 P + \beta_2 Q + \gamma_2 R + \frac{\partial}{\partial y} + \ldots,$$

we get

$$(Y, P) = X + (a_1 - \alpha_1) P + (b_1 - \beta) Q + (c_1 - \gamma_1) R + \beta_2 (Q, P) + \gamma_2 (R, P)$$

$$= X + (a_1 - \alpha_1 + 2\gamma_2) P + (b_1 - \beta_1) Q + (c_1 - \gamma_1 + \beta_2) R;$$

and, similarly,

$$(X, Q) = Y + (a_2 - \alpha_2) P + (b_2 - \beta_2 - 2\gamma_1) Q + (c_2 - \gamma_2 - \alpha_1) R.$$

We now choose the undetermined constants α_1, β_1, γ_1, α_2, β_2, γ_2 so as to make

$$(1) \qquad (Y, P) = X \quad \text{and} \quad (X, Q) = Y.$$

We next suppose ($a_2, b_2, \ldots$ denoting unknown constants) that

$$(Y, Q) = a_2 P + b_2 Q + c_2 R;$$

for obviously (Y, Q) does not involve X, Y, when we express it in terms of X, Y, P, Q, R, a set of five independent operators of the group which is of order five. Similarly we take

$$(X, P) = a_1 P + b_1 Q + c_1 R.$$

We now apply Jacobi's identities to eliminate as far as possible these unknown structure constants of the group.

From

$$(Q, (Y, P)) + (P, (Q, Y)) + (Y, (P, Q)) = 0,$$

$$(Q, (X, P)) + (P, (Q, X)) + (X, (P, Q)) = 0,$$

and from (1) we now have

$$(Y, R) = Y - b_2 R + 2\,c_2 P,$$
$$(X, R) = X + a_1 R - 2 c_1 Q;$$

and from

$$(R, (Y, P)) + (P, (R, Y)) + (Y, (P, R)) = 0,$$

we deduce

$$(R, X) + (P, b_2 R - 2c_2 P - Y) + 2\,(Y, P) = 0;$$

that is, $$2\,c_1 Q - a_1 R + 2\,b_2 P = 0,$$

which, since the operators are independent,

gives $$c_1 = a_2 = b_2 = 0.$$

Similarly we see that $c_2 = a_1 = b_1 = 0$;

and we have now proved that

$$(Y, Q) = 0, \quad (X, P) = 0, \quad (Y, R) = Y, \quad (X, R) = X.$$

In order to complete the structure of the group, we have now only to express the alternant (X, Y) in terms of X, Y, P, Q, R; suppose that

$$(X, Y) = aX + bY + cP + dQ + eR;$$

from $$(P, (X, Y)) + (Y, (P, X)) + (X, (Y, P)) = 0$$

we deduce that $$bX + dR - 2\,eP = 0,$$

and therefore $$b = d = e = 0.$$

Similarly we see that a and c are both zero, and the group has therefore the same structure as the group

$$(2) \qquad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The group (2) and the required group are then simply isomorphic, and the sub-groups of the origin correspond, so that (§ 133) the groups are similar. The only primitive group of the plane of the first class is therefore of the type (2); that is, the type is that of the *special linear group* whose finite equations are

$$x' = ax + by + e, \quad y' = cx + dy + f,$$

where $ad - bc$ is equal to unity.

§ 269. We now have to consider the possible primitive groups of the second class, when the group of the origin contains

$$y\frac{\partial}{\partial x} + \ldots, \quad x\frac{\partial}{\partial y} + \ldots, \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \ldots, \quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \ldots.$$

We have seen that the only operators of the second degree are of the form

$$\xi\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots,$$

where ξ is a linear function; forming the alternant of this with $y\frac{\partial}{\partial x}+\ldots$, we get

$$y\,\xi_1\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots,$$

where ξ_1 is a constant.

Similarly we see that the group must contain

$$y\xi_2\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots.$$

Unless then both ξ_1 and ξ_2 are zero, that is, unless the group contains no operator of the second degree it will contain

$$y\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots.$$

Similarly it will contain

$$x\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots.$$

If the group contains no operator of the second degree it may be proved as before that it is of the type of the *general linear group*

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad x\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\quad x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}.$$

If it does contain an operator of the second degree the group contains the eight operators

$$\frac{\partial}{\partial x}+\ldots,\quad \frac{\partial}{\partial y}+\ldots,\quad y\frac{\partial}{\partial x}+\ldots,\quad x\frac{\partial}{\partial y}+\ldots,\quad x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}+\ldots,$$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\ldots,\quad x\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots,\quad y\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+\ldots.$$

§ 270. Let us denote these operators respectively by

(1) $\qquad X,\ Y,\ P,\ Q,\ R,\ U,\ V,\ W.$

We have at once $\qquad (U, V) = V,$

since the alternant (U, V) being of the second degree cannot involve X, Y, P, Q, or R.

So also $\qquad (U, W) = W$, and $(U, P) = aV + bW$,

where a and b are unknown constants; and if we take instead of P the operator $P - aV - bW$, we shall have

$$(U, P - aV - bW) = 0.$$

Since the lowest terms in $P - aV - bW$ are the same as in P, we may suppose that the operators (1) are such that (U, P) is zero; similarly we may suppose that (U, Q) and (U, R) are zero.

We have

$$(U, X) = -X + aP + bQ + cR + dU + eV + fW,$$

which, by taking a new X with the same initial terms as the original X, is reduced to

$$(U, X) = -X;$$

and similarly $\qquad (U, Y) = -Y.$

Now by a change of coordinates we can transform any linear operator into any other; and in particular we can transform

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dots \quad \text{into} \quad x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}$$

by the transformation formulae

$$x' = x + \xi, \quad y' = y + \eta,$$

where ξ and η are functions of x and y, which, when expanded in power series, begin with terms of the second degree at least.

If then we apply this transformation formula the lowest terms in X, Y, P, Q, R, V, W will not be altered in form, U will become $x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$, and the structure constants will of course be unaltered.

It will now be proved that

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad P = y\frac{\partial}{\partial x}, \quad Q = x\frac{\partial}{\partial y},$$

$$R = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad U = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$V = x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \quad W = y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

Take for instance

$$V = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + \xi^{(3)} \frac{\partial}{\partial x} + \eta^{(3)} \frac{\partial}{\partial y} + \xi^{(4)} \frac{\partial}{\partial x} + \eta^{(4)} \frac{\partial}{\partial y} + \ldots,$$

where $\xi^{(k)}$ denotes a homogeneous function of degree k.

We have

$$(U, V) = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + 2 \left(\xi^{(3)} \frac{\partial}{\partial x} + \eta^{(3)} \frac{\partial}{\partial y}\right) + 3 \left(\xi^{(4)} \frac{\partial}{\partial x} + \eta^{(4)} \frac{\partial}{\partial y}\right) + \ldots,$$

and, as (U, V) is equal to V, we must have

$$\xi^{(3)} \frac{\partial}{\partial x} + \eta^{(3)} \frac{\partial}{\partial y} + 2 \left(\xi^{(4)} \frac{\partial}{\partial x} + \eta^{(4)} \frac{\partial}{\partial y}\right) + \ldots$$

identically zero; that is, $\xi^{(3)}, \eta^{(3)}, \xi^{(4)}, \eta^{(4)}, \ldots$ are all zero, and V is merely $x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$.

Similarly for any other operator; so that this primitive group is of the type

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y},$$

that is, of the type of the projective group of the plane.

There are therefore only *three types of primitive groups in the plane, viz.* (1) *the special linear group;* (2) *the general linear group*; (3) *the general projective group.*

CHAPTER XXII

THE IMPRIMITIVE GROUPS OF THE PLANE

§ 271. We shall now sketch the methods by which the imprimitive groups of the plane may be obtained.

The group being imprimitive, the plane can have an infinity of curves drawn upon it, such that by any operation of the group these curves are only transformed *inter se.*

We therefore choose our coordinates so that these curves will be given by $x =$ constant, and then the linear operators of the imprimitive groups must be of the form

$$\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y},$$

where ξ is a function of x alone.

If the operators of the group are now $X_1, \ldots, X_r$ where

$$X_k = \xi_k\frac{\partial}{\partial x}+\eta_k\frac{\partial}{\partial y}, \qquad (k = 1, \ldots, r),$$

then it is clear that $\quad \xi_1\frac{\partial}{\partial x}, \ldots, \xi_r\frac{\partial}{\partial x}$

must generate a group; and, this being a group in a single variable only, we can, by a change of coordinates (which merely consists in taking as the new variable x' a certain function of the old variable x) reduce ξ_k to be of the form $a_k+b_k x+c_k x^2$ where a_k, b_k, c_k are mere constants. By a change of coordinates the operators of an imprimitive group can therefore be reduced to the form

$$X_k = (a_k+b_k x+c_k x^2)\frac{\partial}{\partial x}+\eta_k\frac{\partial}{\partial y}, \qquad (k = 1, \ldots, r).$$

It then follows that imprimitive groups of the plane can be divided into four classes: the first class will only contain operators in which a_k, b_k, and c_k are zero, that is, they will all be of the form $\eta_k\frac{\partial}{\partial y}$; the second class will contain one operator $\frac{\partial}{\partial x}+\eta_1\frac{\partial}{\partial y}$, while all others will be of the form $\eta_k\frac{\partial}{\partial y}$;

the third will contain the two operators

$$\frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}$$

with others of the form $\eta_k \dfrac{\partial}{\partial y}$; the fourth class will have

$$\frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}, \quad x^2\frac{\partial}{\partial x} + \eta_3 \frac{\partial}{\partial y}$$

with others of the form $\eta_k \dfrac{\partial}{\partial y}$.

When we have found all possible forms of groups of one class, in order to find the forms of groups in the class next in order, we take one of these groups, and add to it the operator which differentiates the higher from the lower class. Applying the conditions for a group, we thus find the form of the operator we have added, and the additional conditions necessary (if any), in order that the group of lower class may thus generate one of higher class; this principle will be sufficiently illustrated in what follows.

§ 272. We have first to find the groups of the form

$$\eta_1 \frac{\partial}{\partial y}, \ldots, \eta_r \frac{\partial}{\partial y}.$$

Since x now occurs merely as a parameter we can, by a transformation of the form

$$x' = x, \quad y' = f(x, y),$$

reduce each of these operators to the form

$$(a_k + \beta_k y + \gamma_k y^2) \frac{\partial}{\partial y},$$

where a_k, β_k, γ_k are functions of the parameter x only; this theorem follows from what we proved as to groups in a single variable.

It may be at once verified that by a transformation of the form

$$y' = \frac{a + \beta y}{\gamma + \delta y},$$

where a, β, γ, δ are functions of x only, any operator

$$(a_k + \beta_k y + \gamma_k y^2) \frac{\partial}{\partial y}$$

is unaltered in form, the functions α_k, β_k, γ_k being transformed into other functions of x. The operators of the group are therefore unaltered in form by any transformation of the given type.

Suppose that for every set of constants $\lambda_1, \ldots, \lambda_r$ the quadratic function of y

$$\lambda_1 \eta_1 + \ldots + \lambda_r \eta_r$$

is a perfect square; we may then assume that

$$\eta_k = a_k (\alpha y + \beta)^2, \qquad (k = 1, \ldots, r),$$

and therefore, if we take

$$y' = \frac{1}{\alpha y + \beta},$$

we may reduce the operators of the group to such a form that y does not occur explicitly in the group at all.

The first type of group that we find in this class is therefore of the form

$$(1) \qquad \left[F_1(x)\frac{\partial}{\partial y}, \ldots, F_r(x)\frac{\partial}{\partial y}\right].$$

Since all the operators are permutable, this group is an Abelian one.

§ 273. We next consider the case where the operators are all of the form

$$(\alpha_k + \beta_k y)\frac{\partial}{\partial y}, \qquad (k=1, \ldots, r),$$

that is, the case where all the functions $\gamma_1, \ldots, \gamma_r$ are zero; we cannot at the same time have all the functions $\beta_1, \ldots, \beta_r$ zero, for then this type of group would reduce to the form just considered.

Suppose therefore that β_1 is not zero, and apply the transformation $y' = \alpha_1 + \beta_1 y$, which will enable us to take one of the operators of the required group to be

$$\beta_1 y \frac{\partial}{\partial y}.$$

Forming the alternant of this with $(\alpha_2 + \beta_2 y)\frac{\partial}{\partial y}$ we find that $\alpha_2 \beta_1 \frac{\partial}{\partial y}$ is an operator of the group. Now if all the functions $\alpha_2, \ldots, \alpha_r$ are zero we can by the transformation $y' = \log y$ reduce the group to the type (1); we therefore assume that α_2 is not zero, and forming the alternant of

$\alpha_2 \beta_1 \frac{\partial}{\partial y}$ and $\beta_1 y \frac{\partial}{\partial y}$ we find that $\alpha_2 \beta_1^2 \frac{\partial}{\partial y}$ is an operator of the group. Similarly we should see that $\alpha_2 \beta_1^3 \frac{\partial}{\partial y}$, $\alpha_2 \beta_1^4 \frac{\partial}{\partial y}, \ldots$ are all operators of the group; and therefore, if the group is to be finite, we must assume β_1 to be a mere constant, and we may take this constant to be unity.

We may similarly show that all the functions $\beta_2, \ldots, \beta_r$ are mere constants; and we thus get the second type of groups in the first class to be

$$(2) \qquad F_1(x)\frac{\partial}{\partial y}, \ldots, F_{r-1}(x)\frac{\partial}{\partial y}, \; y\frac{\partial}{\partial y}.$$

§ 274. We now pass to the case where there is at least one function $\alpha_1 + \beta_1 y + \gamma_1 y^2$ which is not a perfect square and in which γ_1 is not zero.

Let $$\alpha_1 + \beta_1 y + \gamma_1 y^2 \equiv \gamma_1 (y-a)(y-\beta),$$

and apply the transformation $y' = \dfrac{y-a}{y-\beta}$, which gives

$$\alpha_1 + \beta_1 y + \gamma_1 y^2 = \gamma_1 y' \frac{\partial}{\partial y'}.$$

We therefore again assume that the group contains an operator $\beta_1 y \frac{\partial}{\partial y}$; and, if we are not to obtain the type (2) over again, there must be at least one other operator

$$(\alpha_2 + \beta_2 y + \gamma_2 y^2)\frac{\partial}{\partial y}$$

in which γ_2 is not zero.

By a transformation $y' = \gamma_2 y$ we may simplify the discussion by having only to consider the case where γ_2 is unity.

Forming the alternant of $(\alpha_2 + \beta_2 y + y^2)\frac{\partial}{\partial y}$ and $\beta_1 y \frac{\partial}{\partial y}$, we find that $(\beta_1 y^2 - \alpha_2 \beta_1)\frac{\partial}{\partial y}$ is an operator of the required group. Forming the alternant of this again with $\beta_1 y \frac{\partial}{\partial y}$, and so proceeding, we get

$$(\beta_1^2 y^2 + \beta_1^2 \alpha_2)\frac{\partial}{\partial y}, \quad (\beta_1^3 y^2 - \beta_1^3 \alpha_2)\frac{\partial}{\partial y}, \ldots,$$

so that the group would be infinite were not β_1 a mere constant, which we may take to be unity.

The group now contains

$$(y^2-a_2)\frac{\partial}{\partial y} \quad \text{and} \quad (y^2+a_2)\frac{\partial}{\partial y},$$

and therefore $y^2\frac{\partial}{\partial y}$ and $a_2\frac{\partial}{\partial y}$; forming the alternant of these two we see that it contains $a_2 y\frac{\partial}{\partial y}$, so that a_2 is a constant.

The group contains $(a_2+\beta_2 y+y^2)\frac{\partial}{\partial y}$, and therefore also $\beta_2 y\frac{\partial}{\partial y}$, so that β_2 is also a mere constant.

If $(a_3+\beta_3 y+\gamma_3 y^2)\frac{\partial}{\partial y}$ is any other operator we find, by taking its alternant with $y\frac{\partial}{\partial y}$, that the group will contain

$$(a_3+\gamma_3 y^2)\frac{\partial}{\partial y} \quad \text{and} \quad (\gamma_3 y^2-a_3)\frac{\partial}{\partial y},$$

and therefore $a_3\frac{\partial}{\partial y}$, $\gamma_3 y^2\frac{\partial}{\partial y}$, and therefore also $\beta_3 y\frac{\partial}{\partial y}$; and we see as before that a_3, β_3, γ_3 must be mere constants.

If $a_2, \ldots, a_r$ are all zero the group will therefore be of the type

$$\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y},$$

which is but a particular case of (2); but if they are not all zero the group will contain the three independent operators

$$(3) \qquad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y}, \quad y^2\frac{\partial}{\partial y},$$

and no others.

We have now found that all groups in the first class must be of the types (1), (2), or (3).

§ 275. Passing to groups of the second class, and first taking (1) of § 272, we have to find the conditions necessary in order that

$$F_1(x)\frac{\partial}{\partial y}, \ldots, F_r(x)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}$$

may generate a group of order $(r+1)$.

If all the functions $F_1, \ldots, F_r$ vanish identically we can

reduce $\frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ to the form $\frac{\partial}{\partial x}$ by a change of coordinates, and thus obtain the type

$$(4) \qquad \frac{\partial}{\partial x}.$$

If they are not all zero we form the alternant of $\frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ and $F_1(x)\frac{\partial}{\partial y}$, and thus see that the group contains

$$\left(\frac{\partial F_1}{\partial x} - F_1 \frac{\partial \eta}{\partial y}\right)\frac{\partial}{\partial y};$$

$\frac{\partial F_1}{\partial x} - F_1 \frac{\partial \eta}{\partial y}$ must now be dependent on $F_1, \ldots, F_r$, and therefore $\frac{\partial \eta}{\partial y}$ is a function of x alone.

We then take η to be of the form $\alpha y + \beta$, where α and β are functions of x; and it may easily be verified that by a transformation of the form

$$x' = x, \quad y' = y\phi(x) + f(x),$$

we may reduce $\frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ to the form $\frac{\partial}{\partial x'}$ without essentially altering the form of the group

$$(1) \qquad F_1(x)\frac{\partial}{\partial y}, \ldots, F_r(x)\frac{\partial}{\partial y}.$$

We have therefore first of all to see what forms these functions $F_1, \ldots, F_r$ must have in order that (1) and $\frac{\partial}{\partial x}$ may generate a group of order $r+1$.

§ 276. We now make a short digression in order to consider a principle of which much use may be made in the investigation of possible types of finite groups.

If X is any linear operator of the group which we seek, we can by a change of coordinates reduce it to the form $\frac{\partial}{\partial x}$; if then any other operator of the group is

$$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z} + \ldots,$$

we see, by taking its alternant with $\frac{\partial}{\partial x}$, that

$$\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial z} + \ldots$$

is an operator of the group; so also must every linear operator of the form

$$\frac{\partial^k \xi}{\partial x^k}\frac{\partial}{\partial x}+\frac{\partial^k \eta}{\partial x^k}\frac{\partial}{\partial y}+\frac{\partial^k \zeta}{\partial x^k}\frac{\partial}{\partial z}+\dots$$

belong to the group.

Now the group being finite only a certain number of these operators can be independent; and therefore there must be some operator of the form

$$\left(\frac{\partial}{\partial x}-a_1\right)^{m_1}\dots\left(\frac{\partial}{\partial x}-a_k\right)^{m_k},$$

(where $a_1, \dots, a_k$ are constants, depending on the structure constants of the group, and $m_1, \dots, m_k$ are positive integers) which will have the property of annihilating each of the functions $\xi, \eta, \zeta, \dots$.

It follows that

$$\xi = e^{a_1 x}(a_{11}x^{m_1-1}+a_{12}x^{m_1-2}+\dots)$$
$$+e^{a_2 x}(a_{21}x^{m_2-1}+a_{22}x^{m_2-2}+\dots)+\dots,$$

where $a_{ij}, \dots$ denotes a function of the variables not containing x; and that we shall have similar expressions for $\eta, \zeta, \dots$.

Since $\left(\frac{\partial}{\partial x}-a_1\right)\xi\frac{\partial}{\partial x}+\left(\frac{\partial}{\partial x}-a_1\right)\eta\frac{\partial}{\partial y}+\left(\frac{\partial}{\partial x}-a_1\right)\zeta\frac{\partial}{\partial z}+\dots$

is an operator within the group, which will not contain x in a higher power than (m_1-2) in the coefficient of $e^{a_1 x}$, and

$$\left(\frac{\partial}{\partial x}-a_1\right)^2\xi\frac{\partial}{\partial x}+\left(\frac{\partial}{\partial x}-a_1\right)^2\eta\frac{\partial}{\partial y}+\left(\frac{\partial}{\partial x}-a_1\right)^2\zeta\frac{\partial}{\partial z}+\dots$$

is an operator in which x only enters in the power (m_1-3) in the coefficient of $e^{a_1 x}$, and so on, it is not difficult to see that the group must contain the following sets of operators.

Operators in which

$$\xi = e^{a_1 x}a_1, \qquad \eta = e^{a_1 x}b_1, \qquad \zeta = e^{a_1 x}c_1,$$
$$\xi = e^{a_1 x}(a_1 x + a_{11}), \quad \eta = e^{a_1 x}(b_1 x + b_{11}), \quad \zeta = e^{a_1 x}(c_1 x + c_{11}),$$
$$\xi = e^{a_1 x}(a_1 x^2 + 2a_{11}x + a_{12}), \quad \eta = e^{a_1 x}(b_1 x^2 + 2b_{11}x + b_{12}),$$
$$\zeta = e^{a_1 x}(c_1 x^2 + 2c_{11}x + c_{12}),$$

and so on, where the letters $a_1, b_1, c_1, \dots$ all denote functions not containing x.

In addition to these there will be the similar sets of operators corresponding to the roots $a_2, \ldots, a_k$; and every possible linear operator of the group will be dependent on the operators here enumerated.

§ 277. Applying this principle to the problem before us, viz. the determination of the forms of $F_1, \ldots, F_r$ in order that

$$F_1(x)\frac{\partial}{\partial y}, \ldots, F_r(x)\frac{\partial}{\partial y}, \frac{\partial}{\partial x}$$

may be the operators of a finite group, we see that the functions denoted by $a_1, b_1, c_1, \ldots$ are now mere constants; and that the group must therefore be of the form

$$(5)\quad \left[e^{a_k x}\frac{\partial}{\partial y},\ \ xe^{a_k x}\frac{\partial}{\partial y}, \ldots, x^{m_k-1}e^{a_k x}\frac{\partial}{\partial y},\ \ \frac{\partial}{\partial x}\right],\quad (k=1,2,3,\ldots).$$

§ 278. We have now to find what groups in the second class may be generated from

$$F_1(x)\frac{\partial}{\partial y}, \ldots, F_r(x)\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial y},$$

by adding the operator $\quad \dfrac{\partial}{\partial x} + \eta\dfrac{\partial}{\partial y}.$

Forming the alternant of $F_1(x)\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial x} + \eta\dfrac{\partial}{\partial y}$, we see that

$$\left(F_1(x)\frac{\partial \eta}{\partial y} - \frac{\partial F_1(x)}{\partial x}\right)\frac{\partial}{\partial y}$$

is an operator of the group; and therefore

$$\text{(A)}\qquad F_1(x)\frac{\partial \eta}{\partial y} - \frac{\partial F_1(x)}{\partial x} = cy + \sum^{k=r} c_k F_k(x),$$

where $c_1, \ldots, c_r$, and c are absolute constants.

Similarly, by forming the alternant of $y\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial x} + \eta\dfrac{\partial}{\partial y}$, we see that $b, b_1, \ldots, b_r$ being a set of constants

$$\text{(B)}\qquad y\frac{\partial \eta}{\partial y} - \eta = by + \sum^{k=r} b_k F_k(x).$$

From (A) we see that η is of the form $\alpha + \beta y + \gamma y^2$, where

α, β, γ are functions of x only; and from (B) we see, on substituting this value for η, that γ is zero, and

$$\alpha = -\sum^{k=r} b_k F_k(x).$$

Now without loss of generality we can add to $\frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ any operator dependent on

$$F_1(x)\frac{\partial}{\partial y}, \ldots, F_r(x)\frac{\partial}{\partial y};$$

and we may therefore suppose that the form of η is so chosen that both α and γ are zero.

By a transformation of the form

$$x' = x, \quad y' = y\,\phi(x)$$

we may, without essentially altering the form of the other operators of the group, so choose the unknown function $\phi(x)$

that $\qquad \frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y}$ may become $\frac{\partial}{\partial x'}$;

and we may thus reduce the group to one of the type

$$(6) \quad \left[e^{a_k x}\frac{\partial}{\partial y},\ x e^{a_k x}\frac{\partial}{\partial y}, \ldots,\ x^{m_k - 1} e^{a_k x}\frac{\partial}{\partial y},\ y\frac{\partial}{\partial y},\ \frac{\partial}{\partial x}\right],$$
$$(k = 1, 2, 3, \ldots).$$

§ 279. The only type of group in this class remaining to be examined is

$$\frac{\partial}{\partial y},\ y\frac{\partial}{\partial y},\ y^2\frac{\partial}{\partial y},\ \frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}.$$

Forming the alternant $\left(\frac{\partial}{\partial y},\ \frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)$ we see that, there being only four operators in the group,

$$\frac{\partial \eta}{\partial y} = a + 2by + 3cy^2,$$

where a, b, c are mere constants; and therefore

$$\eta = \phi(x) + ay + by^2 + cy^3.$$

Forming the alternants of $\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ with $y\frac{\partial}{\partial y}$ and $y^2\frac{\partial}{\partial y}$

respectively, we see that $\phi(x)$ must be a mere constant, and c must be zero; so that the group reduces to the type

$$(7) \qquad \frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y}, \quad y^2\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}.$$

§ 280. In the third class the groups must contain two operators of the forms $\frac{\partial}{\partial x}+\eta_1\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial x}+\eta_2\frac{\partial}{\partial y}$; and clearly in any group of this class there must be a sub-group containing all the operators of the group except $x\frac{\partial}{\partial x}+\eta_2\frac{\partial}{\partial y}$.

We therefore begin by trying whether from the group

$$\left[e^{a_k x}\frac{\partial}{\partial y}, \quad xe^{a_k x}\frac{\partial}{\partial y}, \ldots, x^{m_k-1}e^{a_k x}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}\right], \quad (k=1, 2, 3, \ldots),$$

we can generate a new group of order one higher, by adding an operator of the form $x\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}$.

Forming the alternant of the new operator with $\frac{\partial}{\partial x}$ we see that $\frac{\partial \eta}{\partial x}$ is a function of x only; and forming its alternant with any other operator of the group we see that $\frac{\partial \eta}{\partial y}$ is a function of x only; and therefore we take

$$\eta = cy+\phi(x)$$

where c is a mere constant.

If we substitute this value of η in $x\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}$, and form the alternant with $x^{m_k-1}e^{a_k x}\frac{\partial}{\partial y}$, we shall find that the group must contain $a_k x^{m_k}e^{a_k x}\frac{\partial}{\partial y}$; and, as x^{m_k-1} is given to be the highest power of x in the coefficient of $e^{a_k x}$, we conclude that a_k must be zero.

The group must therefore be of the form

$$\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y},$$

where $$\eta = cy+\sum^{k=r} c_k x^k+\text{constant};$$

and without loss of generality we may say that

$$\eta = cy + c_r x^r.$$

If c is not equal to r, apply the transformation

$$x' = x, \quad y' = y + \frac{c_r x^r}{c - r},$$

when the group takes the simple form

$$(8) \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}.$$

If c is equal to r it is easily seen that by a transformation of coordinates we may take c_r to be unity, and thus obtain the type

$$(9) \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} + (ry + x^r)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}.$$

§ 281. We should next have to try what groups of the form

$$\left[e^{a_k x}\frac{\partial}{\partial y}, \; xe^{a_k x}\frac{\partial}{\partial y}, \ldots, x^{m_k - 1}e^{a_k x}\frac{\partial}{\partial y}, \; y\frac{\partial}{\partial y}, \; \frac{\partial}{\partial x}, \; x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right], \quad (k = 1, 2, 3, \ldots)$$

can exist; and in much the same way we should see that we may take η to be cxy when c is a constant. If we then apply the transformation

$$x' = x, \quad y' = e^{-cx}y,$$

$\frac{\partial}{\partial x}$ becomes $\frac{\partial}{\partial x'} - cy'\frac{\partial}{\partial y'}$, $y\frac{\partial}{\partial y}$ is unaltered in form, and $x\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$ becomes $x'\frac{\partial}{\partial x'}$, whilst the other operators are not essentially altered in form. If we now apply the same reasoning to this type as we applied to the last, we shall see that a_k must be zero, and that the group takes the form

$$(10) \quad \left[\frac{\partial}{\partial y}, \; x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}, \; y\frac{\partial}{\partial y}, \; \frac{\partial}{\partial x}, \; x\frac{\partial}{\partial x}\right],$$

where $r > 0$.

The other types of group in this class can similarly be found; they are

$$(11) \quad \left[\frac{\partial}{\partial y}, \; y\frac{\partial}{\partial y}, \; y^2\frac{\partial}{\partial y}, \; \frac{\partial}{\partial x}, \; x\frac{\partial}{\partial x}\right];$$

(12) $$\left[\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right];$$

(13) $$\left[\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}\right].$$

§ 282. Passing to types of groups in the fourth class we must take each group from the third class, and see whether we can generate a group of the fourth class by adding to it some operator of the form $x^2\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}$.

Thus it may easily be shown that from

$$\left[\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}+(ry+x^r)\frac{\partial}{\partial y}\right], \text{ where } r>0,$$

a group of the required class cannot be generated. On the other hand, the group

$$\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}+cy\frac{\partial}{\partial y}$$

will lead to two types of group of the fourth class; viz.

(14) $$\left[\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}, \quad 2x\frac{\partial}{\partial x}+(r-1)\,y\frac{\partial}{\partial y}, \quad x^2\frac{\partial}{\partial x}+(r-1)\,xy\frac{\partial}{\partial y}\right],$$

where r is greater than zero; and

(15) $$\left[y\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}\right].$$

The other types of groups in this class are

(16) $$\left[\frac{\partial}{\partial y}, \ x\frac{\partial}{\partial y}, \ldots, x^{r-1}\frac{\partial}{\partial y}, \ y\frac{\partial}{\partial y}, \ \frac{\partial}{\partial x}, \ x\frac{\partial}{\partial x}, \ x^2\frac{\partial}{\partial x}+(r-1)xy\frac{\partial}{\partial y}\right], \quad (r>0);$$

(17) $$\left[\frac{\partial}{\partial y}, \quad y\frac{\partial}{\partial y}, \quad y^2\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x}\right];$$

(18) $$\left[\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \quad x^2\frac{\partial}{\partial x}+(2xy+y^2)\frac{\partial}{\partial y}\right];$$

(19) $$\left[\frac{\partial}{\partial x}, \quad 2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}, \quad x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}\right];$$

(20) $$\left[\frac{\partial}{\partial x},\ \ x\frac{\partial}{\partial x},\ \ x^2\frac{\partial}{\partial x}\right].$$

The methods by which these groups of the fourth class are found does not differ essentially from the methods by which the groups of lower class were found.

§ 283. Every imprimitive group of the plane must belong to one of the types enumerated, but these types are not all mutually exclusive; thus the group

$$\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial y},\quad y^2\frac{\partial}{\partial y}$$

in the first class is similar to the group of the fourth class

$$\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial x},\quad x^2\frac{\partial}{\partial x}.$$

In order to divide the imprimitive groups into mutually exclusive types we examine each of the groups we have found as regards their invariant curve systems. For all the groups the system $x=$ constant is an invariant system, but some of the groups have other invariant curve systems.

We first consider the type (1) and suppose that r is greater than unity; we may then by a transformation of coordinates of the form

$$x' = x,\quad y' = y\,\phi(x)$$

simplify the type so as to be able to assume that two operators of the group are $\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial y}$.

Suppose that for this group $f(x, y) =$ constant is an invariant curve system; we must then have

$$\frac{\partial}{\partial y}f(x, y) = \text{some function of } f(x, y).$$

If this function vanishes identically $f(x, y)$ is a mere function of x, and therefore only gives the known invariant system, $x =$ constant. If, however, the function does not vanish identically the curve system $f(x, y) =$ constant can be thrown into such a form that $\frac{\partial f}{\partial y}$ is unity, and therefore

$$y + f(x) = \text{constant}$$

is an invariant curve system for the group. Applying the

operator $x\dfrac{\partial}{\partial y}$ of the group we must then have

$$x\frac{\partial}{\partial y}\left(y+f(x)\right) = \text{some function of } \left(y+f(x)\right);$$

and as this is impossible we conclude that, if r is greater than unity, (1) cannot have any other invariant curve system than $x = \text{constant}$.

If, however, r is equal to unity, the group is of the type $\dfrac{\partial}{\partial y}$; and admits the ∞^∞ curves $y = f(x)$ as invariant systems, where f is an arbitrary functional symbol.

We next can prove that if the type (2) is of order two, it may be thrown into the form

$$\left[\frac{\partial}{\partial y},\quad y\frac{\partial}{\partial y}\right];$$

and for this group there are two invariant systems, viz. $x = \text{constant}$, and $y = \text{constant}$. If the group is of order greater than two the only invariant system is $x = \text{constant}$.

It will be found that for type (3) there are the invariant systems $x = \text{constant}$, and $y = \text{constant}$.

The type (4) is similar to type (1), when the latter is of order unity.

If the type (5) is of order greater than two, the only invariant system is $x = \text{constant}$. If the group is of order two it can be reduced to one or other of the forms

$$\frac{\partial}{\partial y},\quad \frac{\partial}{\partial x},\quad \text{or}\quad \frac{\partial}{\partial y},\quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y};$$

and for either of these groups there is an infinity of invariant curve systems, viz.

$$ax+by = \text{constant},$$

where a and b are arbitrary constants.

The type (6), if the order is three, can be thrown into the form

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad y\frac{\partial}{\partial y}$$

with the invariant systems $x = \text{constant}$, $y = \text{constant}$; if the order is above the third the only invariant system is $x = \text{constant}$.

The type (7) has the invariant systems $x =$ constant, $y =$ constant.

The type (8), if $r > 1$, has only the invariant system $x =$ constant. If, however, $r = 1$, the type is

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y};$$

and, since the group contains $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, the invariant curve system must be of the form

$$ax + by = \text{constant};$$

if c is equal to unity this system is admitted; but if it is not, the only systems admitted are $x =$ constant and $y =$ constant.

The group (9) has only the invariant system $x =$ constant.

The group (10) has only the invariant system $x =$ constant, if $r > 1$; but, if $r = 1$, it has $x =$ constant, $y =$ constant.

The group (11) has the invariant systems $x =$ constant, $y =$ constant.

The group (12) is similar to one of the cases of (5), viz. the case when (5) can be thrown into the form

$$\frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The group (13) is similar to (2), when (2) is of the second order.

The group (14), when $r > 1$, has only the invariant system $x =$ constant; when $r = 1$, it is

$$\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x}, \quad x^2\frac{\partial}{\partial x},$$

and is similar to (7).

The group (15) has only the invariant system $x =$ constant.

The group (16), when $r > 1$, has only the invariant system $x =$ constant; when $r = 1$ it is similar to (11).

The group (17) has the invariant systems $x =$ constant, $y =$ constant.

The group (18) is similar to

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},$$

and has the invariant systems $x =$ constant, $y =$ constant.

The group (19) has only the invariant system $x =$ constant. The group (20) is similar to (3).

§ 284. We now rearrange the imprimitive groups of the plane into mutually exclusive types and into four new classes, corresponding to the different systems of curves, which are invariant under the operations of the groups. We shall denote the operator $\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ by $\xi p + \eta q$.

In Class I we have the group q for which an invariant system is $y + f(x) =$ constant, where $f(x)$ is any function of x whatever.

In Class II

$$[q, p]; \ [q, xp+yq]; \ [q, p, xp+yq];$$

with the invariant curve systems

$$ax + by = \text{constant},$$

where a and b are any constants.

In Class III

$$[q, yq]; \ [q, yq, y^2 q]; \ [p, q, yq];$$

$[q, yq, y^2 q, p]$; $[q, p, xp+cyq]$, c being a constant not unity;

$$[q, yq, p, xp]; \ [q, yq, y^2 q, p, xp];$$

$$[q, yq, y^2 q, p, xp, x^2 p]; \ [p+q, xp+yq, x^2 p+y^2 q];$$

with the invariant curve systems $x =$ constant, $y =$ constant.

In Class IV

$$[F_1(x) q, \ldots, F_r(x) q], \text{ where } r > 1;$$

$$[F_1(x) q, \ldots, F_r(x), \ yq], \text{ where } r > 1;$$

$$[e^{a_k x} q, \ldots, x^{m_k - 1} e^{a_k x} q, p], \text{ where the order} > 2, \text{ and } k = 1, 2, 3, \ldots;$$

$$[e^{a_k x} q, \ldots, x^{m_k - 1} e^{a_k x} q, yq, p], \text{ where the order} > 3, \text{ and } k = 1, 2, 3, \ldots;$$

$$[q, xq, \ldots, x^{r-1} q, p, xp+cyq], \text{ where } r > 1 \text{ and } c \text{ is a constant};$$

$$[q, xq, \ldots, x^{r-1} q, p, xp+(ry+x^r) q], \text{ where } r > 0;$$

$$[q, xq, \ldots, x^{r-1} q, yq, p, xp], \text{ where } r > 1;$$

$$[q, xq, \ldots, x^{r-1} q, p, 2xp+(r-1) yq, x^2 p+(r-1) xyq],$$
$$\text{where } r > 1;$$

$$[q,\ xq,\ \ldots,\ x^{r-1}q,\ yq,\ p,\ xp,\ x^2p+(r-1)xyq], \text{ where } r>1;$$

$$[yq,\ p,\ xp,\ x^2p+xyq];$$

$$[p,\ 2xp+yq,\ x^2p+xyq];$$

with the invariant curve system $x =$ constant.

It is clear that a group in one class cannot be similar to a group in any other class; and it may easily be seen that in the same class no two similar groups have been enumerated.

Every imprimitive group of the plane must therefore belong to one of these twenty-four mutually exclusive types.

CHAPTER XXIII

THE IRREDUCIBLE CONTACT TRANSFORMATION GROUPS OF THE PLANE

§ 285. We have now found all point groups of the plane, and if we extend these we shall have all the extended point groups; if the groups are only extended to the first order and we apply to them contact transformations we shall have the reducible contact groups of the plane. In this chapter we shall show how the irreducible contact groups of the plane are to be obtained.

It must first be proved that the necessary and sufficient condition that a system of contact operators of the plane may be reducible to mere extended point operators by a contact transformation of the plane is that the operators should leave unaltered an equation system of the form

$$\frac{dx}{a} = \frac{dp}{\beta} = \frac{dy}{ap},$$

where a and β are functions of x, y, p.

Let $\quad f(x, y, p) = \text{constant}, \quad \phi(x, y, p) = \text{constant}$

be integrals of this equation system; then, since

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p + \frac{\partial f}{\partial p}\frac{\beta}{a} = 0 \text{ and } \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} p + \frac{\partial \phi}{\partial p}\frac{\beta}{a} = 0,$$

we see, by eliminating $\frac{\beta}{a}$, that the functions f and ϕ are in involution; we can therefore find a contact transformation

(1) $\quad x' = f(x, y, p), \; y' = \phi(x, y, p), \; p' = \psi(x, y, p)$

which will transform the given equation system into

$$dx' = 0, \; dy' = 0.$$

Now if $$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \pi \frac{\partial}{\partial p}$$

be a contact operator which leaves unaltered the equation system $dx = 0$, $dy = 0$, we see that ξ and η must be functions

not containing p; and therefore the operators, as transformed by (1), will be mere extended point operators. The converse is easily proved; for extended point operators do not alter the equation system $dx = 0$, $dy = 0$; that is, they transform a point M_1 into a point M_1. It follows that if we apply to them a contact transformation the reducible operators will leave unaltered the equation system into which $dx = 0$, $dy = 0$ is transformed; that is, an equation system of the form

$$\frac{dx}{a} = \frac{dp}{\beta} = \frac{dy}{ap}.$$

§ 286. We now take x and z as the coordinates of any point in the plane, and we write y instead of p, when the contact operators of the plane become simply those operators in space x, y, z which do not alter the equation

$$dz - ydx = 0.$$

An irreducible group of contact operators of the plane, when regarded as operators in space, must be transitive. For, suppose the group is intransitive, and $f(x, y, z)$ is an invariant; then the operators of the group do not alter the equations

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0, \quad dz - ydx = 0.$$

They therefore leave unaltered a system of equations of the form

$$\frac{dx}{a} = \frac{dy}{\beta} = \frac{dz}{ay},$$

and therefore may be so reduced as to be mere extended point group operators.

Let $\quad \xi(x, y, z)\frac{\partial}{\partial x} + \eta(x, y, z)\frac{\partial}{\partial y} + \zeta(x, y, z)\frac{\partial}{\partial z},$

or, as we shall write it

$$\xi p + \eta q + \zeta r,$$

be a contact operator of the plane regarded as an operator in space x, y, z; and let W be its characteristic function, so that

$$\xi = \frac{\partial W}{\partial y}, \quad \eta = -\frac{\partial W}{\partial x} - y\frac{\partial W}{\partial z}, \quad \zeta = -W + y\frac{\partial W}{\partial y}.$$

Taking a point of general position as the origin of coordinates, we can arrange the operators of the group into sets

as in § 259. To do this we expand the characteristic function in powers of x, y, z; let $\overline{W}$ be the operator which corresponds to the characteristic function W, that is, let

$$\overline{W}=\frac{\partial W}{\partial y}p-\left(\frac{\partial W}{\partial x}+y\frac{\partial W}{\partial z}\right)q-\left(W-y\frac{\partial W}{\partial y}\right)r.$$

We must, therefore, in order to obtain an operator of degree k, consider the terms in W which are of degrees $(k+1)$ and k. Thus corresponding to $W=-1$ we have $\overline{W}=r$, and corresponding to $W=-x$ we have $\overline{W}=q+xr$; more generally we may express these, and similar results, in the tabular form

$$\begin{array}{l} W= \\ \overline{W}= \end{array}\begin{cases}-1, \\ r\ ,\end{cases}\begin{cases}-x\ , \\ q+xr,\end{cases}\begin{cases}y, \\ p,\end{cases}\begin{cases}-z\ , \\ yq+zr,\end{cases}\begin{cases}-x^2\ , \\ 2xq+x^2r,\end{cases}\begin{cases}xy\ , \\ xp-yq,\end{cases}$$

$$\begin{array}{l} W= \\ \overline{W}= \end{array}\begin{cases}y^2\ , \\ 2yp+y^2r,\end{cases}\begin{cases}-xz\ , \\ (z+xy)q+xzr,\end{cases}\begin{cases}yz\ , \\ zp-y^2q,\end{cases}\begin{cases}-z^2\ . \\ 2yzq+z^2r.\end{cases}$$

This table gives us the operators corresponding to terms in W of the second or lower degrees, and, if required, could easily be extended so as to give the corresponding operators for terms of higher degree. Thus, if $W=a+bx+cxy$, where a, b, c are constants, then

$$\overline{W}=-ar-b(q+xr)+c(xp-yq).$$

It will be noticed that the only terms in W which contribute operators to $\overline{W}$ whose lowest terms are of zero degree are $1, x, y$; and the only terms which contribute operators of the first degree are

$$z,\ x^2,\ xy,\ y^2,\ xz,\ yz.$$

The most general contact operator of the first degree is therefore

$$(1)\quad a_1(yq+zr)+a_2xq+a_3(xp-yq)+a_4yp+a_5zq+a_6zp+\dots,$$

where $a_1, \dots, a_6$ are constants, and the terms indicated by $+\dots$ are of degree higher in x, y, z than those written down.

§ 287. If we have a contact group, and consider the operators of the first degree in the group, we have, by neglecting the terms in such operators indicated by $+\dots$, a group which is linear and homogeneous in x, y, z. From the form given by (1) of § 286 for these operators, we see that the plane $z=0$ is invariant under the operations of this linear group; the straight lines through the origin in this plane are therefore transformed

by the operations of a linear homogeneous group in x, y. Unless, then, this linear group is the general or special linear homogeneous group, it must leave at least one straight line through the origin at rest; and therefore the contact group itself must, when we regard it as a point group in space, leave unaltered at least some ∞^2 curves which pass through ∞^3 points of space; the considerations which enabled us to determine the primitive groups of the plane will render this evident.

Now a contact group with the property of leaving ∞^2 curves at rest has been proved to be reducible; and therefore the linear group must be either the general or special linear group.

The group we are investigating must therefore contain at least the following three operators of the first degree

$$(1) \qquad yp+a_1zp+b_1zq+\dots,$$

$$(2) \qquad xq+a_2zp+b_2zq+\dots,$$

$$(3) \qquad xp-yq+a_3zp+b_3zq+\dots.$$

Since the alternant of the first two of these operators is of the form $xp-yq+a_3zp+b_3zq+\dots$, it will only be necessary to assume that the group contains the first two operators.

From the form of the general contact operator of the first degree ((1) § 286), we see that there cannot be more than six independent operators of the first degree, such that no operator of the second degree is dependent upon them; and since the group is transitive in x, y, z there must be three of zero degree. We have therefore to consider four possible classes of groups; in each there will be the three operators

$$p+\dots,\ q+\dots,\ r+\dots;$$

in Class I there will be three operators of the first degree; in Class II four such operators; in Class III five, and in Class IV there will be six.

§ 288. We first examine the possible forms of irreducible groups in Class I; since the three operators (1), (2), (3) of § 287 must occur there cannot be any operators of the forms

$$zp+\dots,\ zq+\dots,\ \text{or}\ yq+zr+\dots.$$

If we form the alternant of (1) and (2) we get

$$(y+a_1z)q-(x+b_3z)p+\dots;$$

and therefore, adding (3), we see that by the limitation im-

posed on this class we must have (a_3-b_2) zero, and also (b_3+a_1) zero. Similarly, by forming the alternants of (1) and (3), and of (2) and (3) respectively, we see that a_2 and b_1 are both zero.

The operators of the first degree in this class are therefore

$$(x+az)q+\ldots,\ (x+az)p-(y+bz)q+\ldots,\ (y+bz)p+\ldots,$$

where a, b, c are constants; and it will now be shown that there are no operators of the second degree in any group of this class, and therefore no operators of any higher degree.

By the point transformation in space

$$\text{(A)} \qquad x'=x+az,\ y'=y+bz,\ z'=z$$

the operators of zero degree, and of the first degree, can be thrown into the forms

$$p+\ldots,\ q+\ldots,\ r+\ldots,$$
$$xq+\ldots,\ xp-yq+\ldots,\ yp+\ldots.$$

It will be noticed that this transformation is not a contact transformation of the plane.

Suppose now that the group could contain an operator of the second degree

$$\xi p+\eta q+\zeta r+\ldots,$$

where ξ, η, ζ are homogeneous functions of the second degree in x, y, z.

If we form alternants of this operator with $p+\ldots$, $q+\ldots$, $r+\ldots$, respectively, the resulting operators, being of the first degree, must be dependent on $xq+\ldots$, $xp-yq+\ldots$, $yp+\ldots$, and operators of higher degree; and therefore the first derivatives of ξ, η, ζ cannot contain z; it follows that the functions ξ, η, ζ themselves cannot contain z.

Also, since there is no operator of the first degree in which the coefficient of r is not zero, the derivatives $\frac{\partial\zeta}{\partial x}$ and $\frac{\partial\zeta}{\partial y}$ are both zero; and therefore ζ vanishes identically.

If, then, any operator of the second degree is to be found in the group at all it must be

$$\text{(B)} \qquad \xi p+\eta q+\ldots,$$

where ξ and η are homogeneous functions of the second degree in x and y.

There can, however, be no such operator; for we proved in § 267 that the operators

$$p+\ldots,\ q+\ldots,\ xq+\ldots,\ xp-yq+\ldots,\ yp+\ldots$$

could not coexist in any finite group with an operator of the form (B), unless the group also contained the operator of the first degree

$$xp + yq + \ldots;$$

and, as the group we are now considering does not contain this operator, we draw the conclusion that in Class I there can be no operator of the second degree, and therefore none of higher degree.

§ 289. The group has therefore only six operators; for brevity we denote

$$p + \ldots \text{ by } P,\ q + \ldots \text{ by } Q,\ r + \ldots \text{ by } R,$$
$$xq + \ldots \text{ by } X_1,\ xp - yq + \ldots \text{ by } X_2,\ yp + \ldots \text{ by } X_3.$$

Clearly in this group X_1, X_2, X_3 is a sub-group—the group of the origin; its structure is

$$(X_1, X_2) = -2X_1,\ (X_1, X_3) = X_2,\ (X_2, X_3) = -2X_3.$$

We also have

$$\begin{aligned}(X_1, P) &= -Q + a_1 X_1 + b_1 X_2 + c_1 X_3,\\ (X_2, P) &= -P + a_2 X_1 + b_2 X_2 + c_2 X_3,\\ (X_3, P) &= \qquad\quad a_3 X_1 + b_3 X_2 + c_3 X_3,\end{aligned}$$

where $a_1, b_1, c_1, \ldots$ denote constants.

By adding to P and Q properly chosen multiples of X_1, X_2, X_3, we may throw these structure constants into the simple form

$$(X_1, P) = -Q,\ \ (X_2, P) = -P,\ \ (X_3, P) = a_3 X_1.$$

If X, Y, Z are any three linear operators we know that

$$\big(X, (Y, Z)\big) + \big(Y, (Z, X)\big) + \big(Z, (X, Y)\big) = 0;$$

this Jacobian identity may be written in the abbreviated form

$$(X, Y, Z) = 0.$$

From $(X_1, X_2, P) = 0$, we now deduce that $(X_2, Q) = Q$; from $(X_3, X_1, P) = 0$, we similarly obtain $(X_3, Q) = -P$; while from $(X_2, X_3, P) = 0$, we shall find that a_3 is zero.

The alternant (Q, X_1) is dependent on X_1, X_2, X_3; if then

$$(Q, X_1) = aX_1 + bX_2 + cX_3,$$

we deduce from $(X_1, X_3, Q) = 0$ that a and b are zero; while from $(X_2, Q, X_1) = 0$ we shall see that c is zero, and therefore (Q, X_1) is zero.

If we now apply the transformation inverse to (A) of § 288, viz.

$$x = x' + az', \quad y = y' + bz', \quad z = z',$$

we shall bring the operators of the group back again to such a form that they are contact operators of the plane x', z'; and we may therefore say that the group in Class I has the six operators

$$p + \ldots, \quad q + \ldots, \quad r + \ldots,$$

$$(x + az)\, q + \ldots, \quad (x + az)\, p - (y + bz)\, q, \quad (y + bz)\, p + \ldots.$$

If we denote these respectively by P, Q, R, X_1, X_2, X_3, we now know so much of the structure of the group as that

$$(1) \quad \begin{array}{lll} (X_1, X_2) = -2X_1, & (X_1, X_3) = X_2, & (X_2, X_3) = -2X_3, \\ (X_1, P) \ = -Q, & (X_2, P) \ = -P, & (X_3, P) \ = 0, \\ (X_1, Q) \ = 0, & (X_2, Q) \ = 0, & (X_3, Q) \ = 0. \end{array}$$

§ 290. If we now form the alternant of P and Q it will be of the form

$$r + \alpha p + \beta q + \ldots,$$

where α and β are constants. For, if u and v are the characteristic functions of the operators $\overline{u}$ and $\overline{v}$, the characteristic function of the alternant $(\overline{u}, \overline{v})$ is

$$\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial z}\right) - \frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial z}\right) - u\frac{\partial v}{\partial z} + v\frac{\partial u}{\partial z};$$

and, as the lowest terms in the characteristics of $p + \ldots$ and $q + \ldots$ are respectively y and $-x$, the lowest term in the characteristic function of their alternant must be -1, and therefore the lowest terms in the alternant must be of the form $r + \alpha p + \beta q$.

We may then say that

$$(P, Q) = R + \alpha P + \beta Q + \gamma X_1 + \delta X_2 + \epsilon X_3,$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are constants; and we may therefore so choose an operator R as to have $(P, Q) = R$ without altering the structure of the group in so far as it is given by (1) of § 289.

From the identity $(X_1, P, Q) = 0$ we then see that (X_1, R) is zero; and we similarly obtain $(X_2, R) = 0$ and $(X_3, R) = 0$.

We now take

$$(P, R) = a_1 P + b_1 Q + c_1 R + \alpha_1 X_1 + \beta_1 X_2 + \gamma_1 X_3,$$

where $a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1$ are constants.

From $(X_3, P, R) = 0$, we see that a_1, b_1, β_1 are all zero; from $(X_2, P, R) = 0$, we see that c_1 and γ_1 are zero; while from $(X_1, P, R) = 0$, we see that

$$(X_1(P, R)) + (Q, R) = 0.$$

We therefore have

$$(P, R) = aP, \quad (Q, R) = aQ, \quad (P, Q) = R;$$

and from $(P, Q, R) = 0$, we now deduce that a is zero.

The structure of the group is now given by

$$\begin{array}{llll}
 & (P, Q) \;\;= R, & (R, P) \;\;= 0, & (Q, R) \;\;= 0, \\
 & (X_1, P) = -Q, & (X_2, P) = -P, & (X_3, P) = 0, \\
(1) & (X_1, Q) = 0, & (X_2, Q) = Q, & (X_3, Q) = -P, \\
 & (X_1, R) = 0, & (X_2, R) = 0, & (X_3, R) = 0, \\
 & (X_2, X_3) = -2X_3, & (X_3, X_1) = -X_2, & (X_1, X_2) = -2X_1.
\end{array}$$

§ 291. In this group the operators P, Q, R form a simply transitive sub-group of the same structure as the simply transitive group whose operators are

$$p, \quad q + xr, \quad r;$$

it is therefore possible to find a point transformation which will transform P, Q, R to these respective forms.

If we take X_1, X_2, X_3 to be (in the new coordinates thus introduced) respectively

$$\xi_1 p + \eta_1 q + \zeta_1 r, \quad \xi_2 p + \eta_2 q + \zeta_2 r, \quad \xi_3 p + \eta_3 q + \zeta_3 r,$$

then, from the structure constants of the group, we derive a number of equations which these functions $\xi_1, \eta_1, \zeta_1, \ldots$ must satisfy.

It will be at once seen, on forming these equations, that they will be satisfied by taking

$$\xi_1 = 0, \; \eta_1 = x, \; \zeta_1 = \tfrac{1}{2}x^2, \; \xi_2 = x, \; \eta_2 = y, \; \zeta_2 = 0,$$
$$\xi_3 = y, \; \eta_3 = 0, \; \zeta_3 = \tfrac{1}{2}y^2;$$

and therefore a possible form of group is

$$(1) \quad p, \; q + xr, \; r, \; xq + \tfrac{1}{2}x^2 r, \; xp - yq, \; yp + \tfrac{1}{2}y^2 r.$$

Now any group in Class I can be reduced to such a form as to have the structure given by (1) § 290; and for such a group X_1, X_2, X_3 will be the sub-group of the origin. The most general group of the class we seek is therefore simply

isomorphic with (1); and in this isomorphism the groups of the origin correspond, so that (§ 133) we conclude that the most general group is similar to (1); that is, it is reducible to the form (1) by a point transformation in space x, y, z.

§ 292. It must finally be proved that this point transformation is a contact transformation in the plane x, z.

First it may be seen that (1) of § 291 is a contact group, and that it satisfies the condition of irreducibility; we see that all the operators are contact operators, since the corresponding infinitesimal transformations do not alter the equation $dz - y dx = 0$; and we conclude that the group is irreducible because the lowest terms in the operators of the first degree form the special linear homogeneous group (§ 287).

Now suppose that the point transformation, which transforms the general contact group of Class I into (1) of § 291 has transformed the Pfaffian equation $dz - y dx = 0$ into some equation of the form

$$\xi dx + \eta dy + \zeta dz = 0.$$

The group (1) of § 291 must therefore leave unaltered this equation, and also, since the group is a contact one, it must leave unaltered the equation $dz - y dx = 0$; but this would necessitate that (1) of § 291 should leave unaltered a system of the form

$$\frac{dx}{a} = \frac{dy}{\beta} = \frac{dz}{ay},$$

where a and β are functions of x, y, z; and therefore it would be reducible, which we know it is not.

We conclude, therefore, that the only group in Class I is that one which is reducible to

$$p, \quad q + xr, \quad xq + \tfrac{1}{2}x^2 r, \quad xp - yq, \quad yp + \tfrac{1}{2}y^2 r,$$

by a contact transformation of the plane.

§ 293. We shall now briefly consider the groups of irreducible contact transformations of the other classes.

Every such group contains the three operators

$$(1) \qquad yp + a_1 zp + b_1 zq + \ldots,$$

$$(2) \qquad xq + a_2 zp + b_2 zq + \ldots,$$

$$(3) \qquad xp - yq + a_3 zp + b_3 zq + \ldots;$$

and must contain at least one operator of the form

$$(4) \qquad a(xp + yq + 2zr) + bzp + czq + \ldots.$$

If we form the alternants of (1), (2), (3), (4) we see that the group must contain the six operators

$$
\begin{array}{ll}
(1,\ 2) & (y+a_1z)\,q-(x+b_2z)\,p+\dots; \\
(1,\ 3) & (y+a_1z)\,p-b_1zq+(y-b_3z)\,p+\dots; \\
(1,\ 4) & -az\,(a_1p+b_1q)-czp+\dots; \\
(2,\ 3) & -2xq-(b_2+a_3)\,zq+a_2zp+\dots; \\
(2,\ 4) & -az\,(a_2p+b_2q)-bzq+\dots; \\
(3,\ 4) & -az\,(a_3p+b_3q)-bzp+czq+\dots.
\end{array}
$$

Now if the group is of Class III or Class IV it contains at least one operator for which a is zero; and therefore we see from (1, 4), (2, 4), (3, 4) that it must contain $zp+\dots$, and also $zq+\dots$.

If then the group is of Class III, as it can have only five operators of the first degree, its operators must be

$$yp+\dots,\ xq+\dots,\ xp-yq+\dots,\ zp+\dots,\ zq+\dots.$$

If the group is of Class IV it has six operators of the first degree, which must then be

$$yp+\dots,\ xq+\dots,\ xp-yq+\dots,\ xp+yq+2zr\dots,$$
$$zp+\dots,\ zq+\dots.$$

It only remains then to find the operators of the first degree for a group in Class II which can only have four operators of the first degree.

For a group of this class a cannot be zero; for then there would be at least five operators of the first degree, viz. in addition to (1), (2), (3), the operators $zp+\dots$, and $zq+\dots$.

From (2, 3), (3, 1), and (1, 2) we see that, since the group contains (1), (2), (3), it must contain

$$(a_1+b_3)\,zq+(a_3-b_2)\,zp+\dots,\quad 3b_1zq+(a_1+b_3)\,zp+\dots,$$
$$3a_2zp+(b_2-a_3)\,zq+\dots;$$

and therefore, since the group, being in Class II, can contain none of these operators, we must have

$$a_1+b_3=0,\quad a_3-b_2=0,\quad b_1=0,\quad a_2=0.$$

From the equations (1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4) we then deduce that

$$aa_1+c=0,\quad ab_2+b=0,\quad aa_3+b=0,\quad ab_3-c=0;$$

and, since a is not zero, it follows that the operators of the first degree in Class II must be of the form

$$(x+az)\,q+\ldots,\ (x+az)\,p-(y+bz)\,q+\ldots,\ (y+bz)\,p+\ldots,$$
$$xp+yq+2\,zr-azp-bzq,$$

where a and b are some undetermined constants.

§ 294. Having found the initial terms in the operators of the first degree, the methods by which we find the groups in the Classes II, III, and IV are not essentially different from the methods employed in finding the group in Class I, and in finding the primitive groups of the plane; we shall therefore merely state the results which one will arrive at by such an investigation.

Every group of Class II is reducible by a contact transformation to the type

$$p,\ q+xr,\ r,\ xq+\tfrac{1}{2}x^2r,\ xp-yq,\ yp+\tfrac{1}{2}y^2r,\ xp+yq+2\,zr.$$

In the third class no irreducible group can exist.

In Class IV every group is reducible by a contact transformation to the type

$$p,\ \ q+xr,\ \ r,\ \ xq+\tfrac{1}{2}x^2r,\ \ xp-yq,\ \ yp+\tfrac{1}{2}y^2r,$$
$$xp+yq+2\,zr,\ \ (z-xy)\,p-\tfrac{1}{2}y^2q-\tfrac{1}{2}xy^2r,\ \ \tfrac{1}{2}x^2p+zq+xzr,$$
$$(xz-\tfrac{1}{2}x^2y)\,p+(yz-\tfrac{1}{2}xy^2)\,q+(z^2-\tfrac{1}{4}x^2y^2)\,r.$$

There are, therefore, only three types of irreducible contact groups in the plane.

CHAPTER XXIV

THE PRIMITIVE GROUPS OF SPACE

§ 295. It would occupy too much time to attempt to describe all the types of group which may exist in three-dimensional space, and we shall therefore confine our attention to the primitive groups which are the most interesting. It will be shown that there are only eight types of such groups.

The first theorem which it is necessary to establish is that every sub-group of the projective group of the plane must have either an invariant point, an invariant straight line, or an invariant conic.

Suppose that $u=0$ is a curve which *admits* two independent projective operators X and Y, where

$$X=(P_1+xR_1)\frac{\partial}{\partial x}+(Q_1+yR_1)\frac{\partial}{\partial y},$$

$$Y=(P_2+xR_2)\frac{\partial}{\partial x}+(Q_2+yR_2)\frac{\partial}{\partial y},$$

$P_1, Q_1, R_1, P_2, Q_2, R_2$ denoting linear functions of x and y. Then, since all points on the curve $u=0$, must satisfy the equations $Xu=0$, $Yu=0$ these points must also satisfy the equation

$$\begin{vmatrix} P_1+xR_1, & Q_1+yR_1 \\ P_2+xR_2, & Q_2+yR_2 \end{vmatrix}=0,$$

which, it is easily seen, is not a mere identity.

Now this is the equation of a curve of the third degree at most, and, as it contains the curve $u=0$, that curve is an algebraic curve of degree three at the most.

§ 296. We shall now prove that this curve if a cubic must be a degenerate one.

It is easily seen that if A, B, C, D are four points, no three of which are collinear, there is no infinitesimal projective

transformation which can leave all of these points at rest. To prove this, we take any other point P on the plane, then the pencil of four straight lines $A(B, C, D, P)$ must be transformed into a pencil of four other straight lines; and if A, B, C, D were to remain at rest, and P become transformed to P', we should have

$$A(B, C, D, P) = A(B, C, D, P'),$$

so that P' would lie on AP. Similarly it would lie on BP, and therefore P' would coincide with P; that is, every point in the plane would remain at rest, which is of course impossible.

Let A be one of the points of inflexion which every cubic must have: if the cubic admits any projective group the group must leave A at rest; for an inflexion can only be transformed to an inflexion, and therefore if A did not remain at rest there would be an infinity of inflexions.

If the cubic has no double point it must have nine points of inflexion; and at least four of these points are such that no three of them are collinear. A non-singular cubic cannot therefore admit a projective group; for the group would then leave four non-collinear points at rest, which is impossible.

We conclude, therefore, that the cubic has a double point. Suppose that it contains one double point and no cusp; it has then three points of inflexion, and these points, together with the double point, must remain at rest under the operations of the group. But if a point A and three points B, C, D on a straight line not passing through A remain at rest, the only projective transformation which the figure could admit would be a perspective one with A as centre and BCD as axis of perspective.

An infinitesimal projective transformation cannot therefore transform the cubic into itself; for, if P is any point on the curve and A the double point, P would have to be transformed to a near point P' on the line AP; and P' could not be on the curve, since AP only intersects the cubic on A and P.

Suppose now that the cubic has one cusp only; since by hypothesis the cubic admits at least two infinitesimal transformations, there must be at least one infinitesimal transformation which will not alter the position of some arbitrarily assigned point P on the cubic. From P draw the tangent PQ which touches the cubic at a point Q distinct from P: there will now be four points, viz. P, Q, the point of inflexion, and the cusp which will not be altered by the projective

infinitesimal transformations admitted both by the point P and the cubic itself. As we can so choose P that no three of these points are collinear, we must conclude that the cubic cannot be a proper one.

Since the cubic must be degenerate we conclude that the only curves, which could admit a projective group with at least two operators, are straight lines or conics.

§ 297. Any sub-group of the general projective group of the plane must be either primitive or imprimitive; we first take the case where it is primitive, and therefore of one of the two following types:

$$p,\ q,\ xq,\ xp-yq,\ yp,\ xp+yq;$$

$$p,\ q,\ xq,\ xp-yq,\ yp.$$

The first of these is the general linear group

$$x' = a_1x + b_1y + c_1, \quad y' = a_2x + b_2y + c_2;$$

and it is clear that by any operation of this group a point at infinity will be transformed to a point at infinity; and therefore the group leaves the line at infinity at rest. The second group, being a sub-group of the first, must therefore also leave the line at infinity at rest.

It now remains to prove that every imprimitive projective group of the plane will leave either a point, a line, or a conic at rest.

First we take the case where the group is at least of the third order. From the imprimitive property of the group we know there is an infinity of curves forming an invariant system. If we take any one of these curves there must be at least two infinitesimal transformations of the group which it will admit; for there are at least two such transformations which will not transform any chosen point on the curve from off the curve. Each of these curves must therefore, since the group is projective, be either a conic or a straight line.

If the invariant system of ∞^1 curves are conics, the five coordinates of the conic must be connected by four equations, and therefore the system of conics must have an envelope. This envelope may consist of mere isolated points; thus the envelope of conics of the system $u + kv = 0$, where k is a parameter, consists of the four points of intersection of the two conics $u = 0$ and $v = 0$.

Similarly, if the invariant system of ∞^1 curves are straight lines, they must have an envelope.

Now the envelope is invariant under all the transformations

of the group; and, if it does not consist of a mere set of isolated points, it must therefore, by what we have proved, be either a straight line or a conic.

A sub-group of the general projective group, if of at least the third order, will therefore leave at rest either a point, a line, or a conic.

We now suppose the sub-group to be of order two; and take X_1 and X_2 to be its operators; we have

$$(X_1, X_2) = aX_1 + bX_2$$

where a and b are constants; and therefore if we take as the operators of the group X_1 and $aX_1 + bX_2$, we see that the group must have the structure

$$(X_1, X_2) = bX_2.$$

If b is not zero, by taking the fundamental operators of the group (i.e. those in terms of which the others are to be expressed) to be $\frac{1}{b} X_1$ and X_2, we have the structure

$$(X_1, X_2) = X_2;$$

if, however, b is zero the structure is given by

$$(X_1, X_2) = 0.$$

If the group is intransitive there will be an infinity of invariant curves; and, by what we have proved, these must be straight lines or curves. If on the other hand the group is transitive we throw X_2 into the form $\frac{\partial}{\partial x}$; and then we may take X_1 in the form $x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, if the structure is given by $(X_1, X_2) = X_2$; if the operators are permutable, we take X_1 in the form $\frac{\partial}{\partial y}$.

In either case the line at infinity is invariant under the operations of the group; and therefore returning to the original variables *some* curve admits two infinitesimal projective transformations, and therefore must be either a straight line or conic.

Finally if the projective group contains only one operator, let it be

$$\left(e_1 + e_2 x + e_3 y + x(e_4 x + e_5 y)\right) p + \left(e_6 + e_7 x + e_8 y + y(e_4 x + e_5 y)\right) q.$$

The condition that the straight line

$$\lambda x + \mu y + \nu = 0$$

may be invariant requires it to coincide with

$$\lambda\big(e_1+e_2x+e_3y+x(e_4x+e_5y)\big)$$
$$+\mu\big(e_6+e_7x+e_8y+y(e_4x+e_5y)\big)=0.$$

The equations therefore to determine λ, μ, ν are

$$\lambda e_2+\mu e_7-\nu e_4=k\lambda,\quad \lambda e_3+\mu e_8-\nu e_5=k\mu,\quad \lambda e_1+\mu e_6=k\nu,$$

where k is to be determined by

$$\begin{vmatrix} e_2-k, & e_7\;, & e_4 \\ e_3\;, & e_8-k, & e_5 \\ e_1\;, & e_6\;, & k \end{vmatrix}=0;$$

and there is therefore at least one straight line which the group leaves at rest.

In every case, therefore, a sub-group of the general projective group of the plane must leave at rest either a point, a straight line, or a conic.

§ 298. We now proceed to show how the primitive groups of space are to be obtained. We take as origin a point of general position, and arrange the operators of the group according to degree, as in § 259.

There will be three operators of zero degree

$$p+\ldots,\quad q+\ldots,\quad r+\ldots,$$

where we write p for $\frac{\partial}{\partial x}$, q for $\frac{\partial}{\partial y}$, r for $\frac{\partial}{\partial z}$; and a number of operators of the first degree which cannot exceed nine. Let the operators of the first degree be $X_1, X_2, \ldots$ where

$$X_k=(a_{k1}x+a_{k2}y+a_{k3}z)\,p+(b_{k1}x+b_{k2}y+b_{k3}z)\,q$$
$$+(c_{k1}x+c_{k2}y+c_{k3}z)r+\ldots,$$

and $a_{k1}, \ldots, b_{k1}, \ldots, c_{k1}, \ldots$ denote constants.

If we put $x=uz'$, $y=vz'$, $z=z'$, then in the new variables the terms of lowest degree in X_k are transformed into

$$\big(a_{k1}u+a_{k2}v+a_{k3}-(c_{k1}u+c_{k2}v+c_{k3})u\big)\frac{\partial}{\partial u}$$
$$+\big(b_{k1}u+b_{k2}v+b_{k3}-(c_{k1}u+c_{k2}v+c_{k3})v\big)\frac{\partial}{\partial v}$$
$$+(c_{k1}u+c_{k2}v+c_{k3})z'\frac{\partial}{\partial z'}.$$

If we now regard u, v as the line coordinates of straight lines through the origin, we see that the ∞^2 linear elements through the origin are transformed by the group of the origin, in exactly the same way as the straight lines u, v are transformed by Y_1, Y_2, where

$$Y_k = \left(a_{k1}u + a_{k2}v + a_{k3} - (c_{k1}u + c_{k2}v + c_{k3})u\right)\frac{\partial}{\partial u} + \left(b_{k1}u + b_{k2}v + b_{k3} - (c_{k1}u + c_{k2}v + c_{k3})v\right)\frac{\partial}{\partial v}.$$

The linear operators $Y_1, Y_2, \ldots$ are now the operators of a projective group in the variables u, v, and there cannot be more than eight independent operators in such a group.

If there are eight independent operators $Y_1, \ldots, Y_8$ the group is the general projective one

$$u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial v}, \quad v\frac{\partial}{\partial u}, \quad v\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial u},$$
$$\frac{\partial}{\partial v}, \quad u^2\frac{\partial}{\partial u} + uv\frac{\partial}{\partial v}, \quad uv\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v};$$

and the terms of lowest degree in $X_1, \ldots, X_8$ are the terms of the special linear homogeneous group

$$zp, \quad zq, \quad xq, \quad xp - zr, \quad yq - zr, \quad yp, \quad xr, \quad yr.$$

It may be proved by the method of Chapter XXI that in this case the primitive group we seek must be one of the following three:—

The general projective group of space

$$(1) \quad \begin{matrix} [p, q, r, \quad xp, yp, zp, \quad xq, yq, zq, \quad xr, yr, zr, \\ x^2p + xyq + xzr, \quad xyp + y^2q + yzr, \quad xzp + yzq + z^2r]; \end{matrix}$$

the general linear group

$$(2) \quad [p, q, r, \quad xp, yp, zp, \quad xq, yq, zq, \quad xr, yr, zr];$$

the special linear group

$$(3) \quad [p, q, r, \quad xq, xp - yq, yp, \quad zp, zq, \quad xp - zr, xr, yr].$$

§ 299. If $Y_1, Y_2, \ldots$ are not the operators of the general projective group they must form a sub-group of it; and must therefore have the property of leaving at rest either a point, a straight line, or a conic.

They cannot leave any point at rest; for, if they did, the group of the origin, viz. $X_1, X_2, \ldots$ and the operators of higher

degree, would leave at rest a linear element through the origin, and therefore the group would not be a primitive one.

Suppose that $Y_1, Y_2, \ldots$ have as invariant a straight line, then the primitive group we are seeking must have an invariant equation of the form

$$a\,dx+\beta dy+\gamma dz = 0$$

where a, β, γ are functions of x, y, z.

By a change of variables we can reduce this equation to the form

$$dz-y\,dx = 0\,*,$$

and the group we seek must therefore in the new variables be a contact group in the plane xz.

If this contact group were reducible, it would have an invariant equation system of the form

$$\frac{dx}{a}=\frac{dy}{\beta}=\frac{dz}{a\,y};$$

and therefore, regarded as a point group in space, could not be primitive.

Since then it must be irreducible, it can by a contact transformation of the plane be reduced to one of the three forms:

(1) $p,\ q+xr,\ r,\ xq+\frac{1}{2}x^2r,\ xp-yq,\ yp+\frac{1}{2}y^2r;$

(2) $p,\ q+xr,\ r,\ xy+\frac{1}{2}x^2r,\ xp-yq,\ yp+\frac{1}{2}y^2r,\ xp+yq+2zr;$

(3) $$\begin{gathered} p,\ q+xr,\ r,\ xq+\tfrac{1}{2}x^2r,\ xp-yq,\ yp+\tfrac{1}{2}y^2r,\ xp+yq+2zr, \\ (z-xy)\,p-\tfrac{1}{2}y^2q-\tfrac{1}{2}xy^2r,\ \tfrac{1}{2}x^2p+zq+xzr, \\ (xz-\tfrac{1}{2}x^2y)\,p+(yz-\tfrac{1}{2}xy^2)\,q+(z^2-\tfrac{1}{4}x^2y^2)\,r. \end{gathered}$$

If a group is imprimitive, it must be admitted by some equation of the form

(4) $$\xi p+\eta q+\zeta r = 0.$$

Now if for a transformation of the form

(5) $$x'=f(x,y),\quad y'=\phi(x,y),\quad z'=\psi(x,y,z)$$

the equation (4) is invariant, then for the same transformation the equation

$$\xi p+\eta q = 0$$

must be an invariant one.

The group (1) can only be admitted by (4), if ξ, η, ζ do not

* It could not reduce to the form $dz = 0$, for then the group would be imprimitive.

contain x or z; for only equations of this form could admit the operators p and r. Again it is clear that every transformation of (1) is of the form (5), and therefore

$$\xi p+\eta q=0$$

must admit the group

$$p,\ q,\ xq,\ xp-yq,\ yp,$$

formed by omitting the parts of the operators involving r.

This group, however, in x, y is primitive, and cannot be admitted by an equation of the form $\xi p+\eta q=0$; and therefore we conclude that the only equation which could admit (1) is the equation $r=0$.

It can be at once verified that this equation admits both the group (1) and the group (2), so that these groups are imprimitive.

If the group (3) is admitted by an equation of the form

$$(4) \qquad \xi p+\eta q+\zeta r;$$

then, since (1) is a sub-group of (3), the group (1) must also have the equation (4) as an invariant one; from what we have proved therefore, ξ and η must both vanish identically, and we have only to try whether $r=0$ admits the group (3).

Now it can be at once verified that it does not do so; so that (3) is the only primitive group of space obtained from the supposition that $Y_1, Y_2, \ldots$ have as invariant a straight line.

§ 300. If we transform to the variables

$$y=y'^2,\quad x=\frac{x'}{y'},\quad z=z',$$

then in the new variables the Pfaffian equation

$$dz-ydx=0 \text{ becomes } dz'-y'dx'+x'dy'=0;$$

and we have the primitive group of space x, y, z,

$$(1)\quad \begin{array}{l} p-yr,\ q+xr,\ r,\ xq,\ xp-yq,\ yp,\ xp+yq+2zr, \\ zp-y(xp+yq+zr),\ zq+x(xp+yq+zr),\ z(xp+yq+zr), \end{array}$$

characterized by the property of leaving unaltered the equation

$$dz-ydx+xdy=0,$$

and transforming the straight lines of this linear complex *inter se*.

§ 301. We have now only to consider the case where $Y_1, Y_2, \ldots$ has an invariant conic which does not break up into straight lines.

By a projective transformation any conic can be reduced to the form
$$x^2+y^2+1 = 0;$$
and we need therefore only consider the projective group which such a conic can admit.

If the conic admits
$$\left(e_1+e_2x+e_3y+x(e_4x+e_5y)\right)p+\left(e_6+e_7x+e_8y+y(e_4x+e_5y)\right)q,$$
we must have
$$e_3+e_7 = 0,\ e_2=0,\ e_8 = 0,\ e_1-e_4 = 0,\ e_5-e_6 = 0;$$
and therefore the operator must be of the form
$$e_1X+e_2Y+e_3Z,$$
where $X = yp-xq,\ Y=(1+x^2)p+xyq,\ Z = xyp+(1+y^2)q.$

The operators $Y_1, Y_2, \ldots$ must therefore be the operators of the group X, Y, Z with the structure
$$(Y, Z) = X,\ \ (Z, X) = Y,\ \ (X, Y) = Z,$$
or of one of its sub-groups.

If the sub-group is of order one we have proved that it leaves a straight line at rest, and therefore comes under the case already considered.

Next we take the case where the sub-group is of order two, and we take its operators to be
$$e_1X+e_2Y+e_3Z \ \text{ and } \ \epsilon_1X+\epsilon_2Y+\epsilon_3Z.$$

Since the alternant of these two operators must be dependent on them we must have
$$(e_1X+e_2Y+e_3Z,\ \ \epsilon_1X+\epsilon_2Y+\epsilon_3Z)$$
$$= p\,(e_1X+e_2Y+e_3Z)+q\,(\epsilon_1X+\epsilon_2Y+\epsilon_3Z);$$
and therefore, since the alternant is easily proved equivalent to
$$(e_2\epsilon_3-e_3\epsilon_2)X+(e_3\epsilon_1-e_1\epsilon_3)Y+(e_1\epsilon_2-e_2\epsilon_1)Z,$$
we have
$$\begin{vmatrix} e_2\epsilon_3-e_3\epsilon_2, & e_3\epsilon_1-e_1\epsilon_3, & e_1\epsilon_2-e_2\epsilon_1 \\ e_1\quad, & e_2\quad, & e_3 \\ \epsilon_1\quad, & \epsilon_2\quad, & \epsilon_3 \end{vmatrix} = 0;$$

that is, $(e_2\epsilon_3-e_3\epsilon_2)^2+(e_3\epsilon_1-e_1\epsilon_3)^2+(e_1\epsilon_2-e_2\epsilon_1)^2=0.$

If we choose λ, μ, ν to satisfy the equations

$$\lambda e_1+\mu e_2+\nu e_3=0, \quad \lambda\epsilon_1+\mu\epsilon_2+\nu\epsilon_3,$$

it can be at once verified that the straight line

$$\lambda=\mu y-\nu x$$

admits this sub-group, so that this also falls under the case already considered.

We have therefore only to consider the case where the group Y_1, Y_2, ... is of the third order.

§ 302. We must now find the form of a group in x, y, z which is of at least the sixth order, with three operators of zero degree, and at least three of the first degree, and with the property of having an invariant equation of the form

$$(1) \quad a dx^2+b dy^2+c dz^2+2f dy dz+2g dz dx+2h dx dy=0,$$

where a, b, c, f, g, h are functions of x, y, z such that the discriminant

$$abc+2fgh-af^2-bg^2-ch^2$$

is not zero.

The equation (1) is not altered in form by any point transformation, and it may easily be proved that by a suitably chosen transformation we may reduce it to the form

$$(2) \qquad a dx^2+b dy^2+c dz^2=0.$$

The origin being a point of general position, and the discriminant not being zero, we know that if we expand the functions a, b, c in powers of the variables the lowest terms will be of degree zero; and by a linear transformation we may take these lowest terms each to be unity. We must now find all possible forms of primitive groups of order not less than six which the Mongian equation (2) can admit.

Arranging the operators according to degree, as in § 259, we shall first prove that the group cannot contain an operator of degree three, and therefore none of higher degree.

If the equation admits the operator

$$X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z},$$

we must have, for all values of x, y, z, dx, dy, dz, satisfying (2),

$$2a(\xi_1 dx+\xi_2 dy+\xi_3 dz)dx+2b(\eta_1 dx+\eta_2 dy+\eta_3 dz)dy$$
$$+2c(\zeta_1 dx+\zeta_2 dy+\zeta_3 dz)dz+Xa\,.\,dx^2+Xb\,.\,dy^2+Xc\,.\,dz^2=0,$$

where suffixes are used to denote partial derivatives.

It therefore follows that we must have

$$b\eta_3 + c\zeta_2 = 0,\ \ c\zeta_1 + a\xi_3 = 0,\ \ a\xi_2 + b\eta_1 = 0,$$

and, if ρ denotes some undetermined factor,

$$2a\xi_1 + Xa = \rho a,\ \ 2b\eta_2 + Xb = \rho b,\ \ 2c\zeta_3 + Xc = \rho c.$$

We now suppose X to be an operator of the third degree of which the terms of lowest degree are

$$\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z},$$

so that
$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z} + \dots.$$

The equations satisfied by ξ, η, ζ are now

$$\begin{aligned} &\eta_3 + \zeta_2 = 0, \quad \zeta_1 + \xi_3 = 0, \quad \xi_2 + \eta_1 = 0, \\ &2a\xi_1 \ = \rho a, \ \ 2b\eta_2 \ = \rho b, \ \ 2c\zeta_3 \ = \rho c, \end{aligned}$$

since we may neglect Xa, Xb, Xc, as containing no terms of degree less than three, while the derivatives of ξ, η, ζ only contain terms of the second degree.

These equations can be written

$$\eta_3 + \zeta_2 = 0,\ \ \zeta_1 + \xi_3 = 0,\ \ \xi_2 + \eta_1 = 0,\ \ \xi_1 = \eta_2 = \zeta_3;$$

and we have proved in Chapter II, § 35, that no values of ξ, η, ζ of the third degree can be found to satisfy these equations; we therefore conclude that the group cannot contain any operator of the third degree.

§ 303. Still making use of the results of Chapter II, we shall see that the only possible operators of the second degree are dependent upon

$$(1)\quad (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2zx\frac{\partial}{\partial z} + \dots,$$

$$(2)\quad 2xy\frac{\partial}{\partial x} + (y^2 - z^2 - x^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + \dots,$$

$$(3)\quad 2zx\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} + \dots.$$

Similarly we see that the only possible operators of the first degree are dependent upon

$$(4)\quad y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + \dots,$$

$$(5) \quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}+\ldots,$$

$$(6) \quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}+\ldots,$$

$$(7) \quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\ldots;$$

and therefore the group is of the tenth order at highest.

We next see, as in § 264, by aid of the isomorphic group $Y_1, Y_2, \ldots$ in the variables u, v, that there must be three operators of the first degree at least, viz.

$$y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}+e\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)+\ldots,$$

$$z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}+e\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)+\ldots,$$

$$x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}+e\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)+\ldots,$$

where e is a constant.

If we form the alternants of these three we see that, unless e is zero, the group must also contain

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\ldots,$$

and therefore the group must contain (4), (5), (6), and may also contain (7).

If we denote by $\bar{1}$ the operator (1) and so on, we see that $\bar{1}$, $\bar{2}$, and $\bar{3}$ are commutative; and that

$$(\bar{1}, \bar{4}) = 0, \quad (\bar{1}, \bar{5}) = -\bar{3}, \quad (\bar{1}, \bar{6}) = \bar{2}, \quad (\bar{2}, \bar{5}) = 0, \quad (\bar{2}, \bar{4}) = \bar{3},$$
$$(\bar{2}, \bar{6}) = -\bar{1}, \quad (\bar{3}, \bar{6}) = 0, \quad (\bar{3}, \bar{5}) = \bar{1}, \quad (\bar{3}, \bar{4}) = -\bar{2}.$$

From these identities we see that if the group admits any operator of the second degree, viz. (1), (2) or (3), it must admit all three.

We first consider the case where the group admits no operator of the second degree, and not (7), but only (4), (5), (6) in addition to the three of zero degree.

If we denote $\bar{4}$ by X, $\bar{5}$ by Y, $\bar{6}$ by Z, and the three operators of zero degree,

$$p+\ldots, \quad q+\ldots, \quad r+\ldots,$$

by P, Q, R respectively, we have

$$(Y, Z) = -X, \quad (Z, X) = -Y, \quad (X, Y) = -Z.$$

We also have, since X, Y, Z, P, Q, R generate a group,

$$(P, X) = a_1 X + b_1 Y + c_1 Z, \quad (P, Y) = -R + a_2 X + b_2 Y + c_2 Z,$$
$$(P, Z) = Q + a_3 X + b_3 Y + c_3 Z,$$

where a_1, b_1, ... denote structure constants; if we add to P, Q, R operators dependent on X, Y, Z, we may throw these identities into the simpler forms

$$(P, X) = aX, \quad (P, Y) = -R, \quad (P, Z) = Q,$$

where a is some constant.

From the Jacobian identity

$$(P, (X, Y)) + (Y, (P, X)) + (X, (Y, P)) = 0,$$

which we now write in the form $(P, X, Y) = 0$, as we shall have occasion to employ it often, we deduce

$$(R, X) = -Q + aZ;$$

while, from $(P, X, Z) = 0$, we have

$$(Q, X) = R + aY;$$

and, from $(P, Y, Z) = 0$, we have

$$(R, Z) + (Q, Y) = aX.$$

We now have $(Q, Z) = -P + a_1 X + b_1 Y + c_1 Z$,

and deduce, from $(Q, X, Z) = 0$, that

$$(Q, Y) - (R, Z) = c_1 Y - b_1 Z; \text{ and therefore}$$
$$2(Q, Y) = aX + c_1 Y - b_1 Z, \quad 2(R, Z) = aX - c_1 Y + b_1 Z.$$

From $(Q, Y, Z) = 0$, we then conclude that a, a_1, and c_1 are zero; and have so far determined the structure of the group that we may say that

$$(P, X) = 0, \quad (P, Y) = -R, \quad (P, Z) = Q, \quad (Q, X) = R,$$
$$(Q, Y) = -bZ, \quad (Q, Z) = -P + 2bY, \quad (R, X) = -Q, \quad (R, Z) = bZ.$$

From $(Q, X, Y) = 0$, we now see that

$$(R, Y) = P - bY;$$

and, from $(R, X, Y) = 0$, we see that b is also zero.

Suppose that

$$(P, Q) = a_1 P + b_1 Q + c_1 R + \lambda X + \mu Y + \nu Z;$$

we then see from $(P, Q, X) = 0$, and from $(P, Q, Y) = 0$, that

$$(R, P) = c_1 Q - b_1 P - \mu Z + \nu Y,$$
$$(Q, R) = c_1 P - a_1 R - \lambda Z + \nu X;$$

and, from $(P, Q, Z) = 0$, we conclude that a_1, b_1, λ, μ are all zero, and therefore

$$(P, Q) = c_1 R + \nu Z, \quad (Q, R) = c_1 P + \nu X, \quad (R, P) = c_1 Q + \nu Y.$$

If we now take as the operators of the group instead of P the operator $P + eX$, instead of Q the operator $Q + eY$, and instead of R the operator $R + eZ$, it is seen that the only structure constants which are changed are c_1 and ν which become respectively $c_1 - 2e$ and $\nu - c_1 e + e^2$. By properly choosing e we can therefore throw the structure of the group into the form

$$\begin{array}{lllll} (Y,Z)=-X, & (Z,X)=-Y, & (X,Y)=-Z, & (P,X)=0, & (Q,Y)=0, \\ (R,Z)=0, & (P,Y)=-R, & (P,Z)=Q, & (Q,X)=R, & (Q,Z)=-P, \\ (R,X)=-Q, & (R,Y)=P, & (Q,R)=cP, & (R,P)=cQ, & (P,Q)=cR. \end{array}$$

§ 304. Two cases now present themselves according as c is, or is not, equal to zero.

First we take the case where c is zero.

P, Q, R now form a simply transitive Abelian sub-group. By a point transformation we can therefore reduce P, Q, R to the forms $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ respectively; suppose that

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z},$$

where in ξ, η, ζ the lowest terms, when expanded in powers of x, y, z, are of the first degree. From

$$(P, X) = 0, \quad (Q, X) = R, \quad (R, X) = -Q,$$

we see that (denoting partial differentiation with respect to x, y, z by the suffixes 1, 2, 3, respectively)

$$\xi_1 = \eta_1 = \zeta_1 = 0, \quad \xi_2 = \eta_2 = 0, \quad \zeta_2 = 1, \quad \xi_3 = \zeta_3 = 0, \quad \eta_3 = -1;$$

and therefore $$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Similarly we see that $Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$ and $Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$; and therefore the group is simply the group of movements in ordinary space; and the invariant Mongian equation is

$$dx^2 + dy^2 + dz^2 = 0.$$

Next we take the case where c is not zero; and we choose, as the fundamental operators of the group,

$$P,\ Q,\ R,\ X-cP,\ Y-cQ,\ Z-cR,$$

which we may denote by

$$P,\ Q,\ R,\ P',\ Q',\ R'.$$

The structure is now given by

$$(P,\ Q) = -R,\ \ (Q,\ R) = -P,\ \ (R,\ P) = -Q,$$
$$(P',Q') = -R',\ \ (Q',R') = -P',\ \ (R',P') = -Q',$$

while each of the operators P, Q, R are commutative with each of the operators P', Q', R'.

We may also rearrange these operators, taking

$$U = -P + iR,\ \ V = iQ,\ \ W = -P - iR,$$
$$U' = -P' + iR',\ \ V' = iQ',\ \ W' = -P' - iR',$$

where i is the symbol for $\sqrt{-1}$; the group is now the direct product of two simply transitive reciprocal groups.

Since U, V, W is simply transitive, and has the same structure as

$$q + xr,\ \ yq + zr,\ \ (xy-z)p + y^2q + yzr,$$

it may be transformed into the latter when U', V', W' will be transformed into

$$p + yr,\ \ xp + zr,\ \ x^2p + (xy-z)q + zxr.$$

It will be noticed that in this form the origin is no longer a point of general position; and it may at once be verified that in this form the group has the invariant Mongian equation

$$dz^2 + y^2dx^2 + x^2dy^2 + (4z - 2xy)\,dx\,dy - 2x\,dy\,dz - 2y\,dz\,dx = 0.$$

This group, which is admitted by the quadric $z - xy = 0$, is the group of movements in non-Euclidean space.

§ 305. If we were to consider the case of a group containing no operators of the second, but four of the first degree, and three of zero degree, we should similarly obtain the group of order seven consisting of movements in Euclidean space and uniform expansion, viz.

$$p,\ q,\ r,\ yr - zq,\ zp - xr,\ xq - yp,\ xp + yq + zr.$$

Finally, if we were to consider the group containing three

operators of the second degree, we should find that there must be four operators of the first degree in the group, as well as three of zero degree; and should arrive at the conformal group in three-dimensional space, consisting of movements in Euclidean space, uniform expansion and inversions, viz. the group

$$(1) \quad [p,\ q,\ r,\ xq-yp,\ yr-zq,\ zp-xr,\ U,\ 2xU-Sp,\ 2yU-Sq,\ 2zU-Sr],$$

where $U = xp+yq+zr$ and $S = x^2+y^2+z^2$.

This group has the property of being the most general group for which the equation

$$dx^2+dy^2+dz^2 = 0$$

is an invariant.

By the operations of this group any sphere is transformed into a sphere, and in particular any point sphere

$$(x-a)^2+(y-b)^2+(z-c)^2 = 0$$

is transformed into some other point sphere. If, therefore, we apply the contact transformation with the generating equations

$$x'+iz'+xy'-z = 0, \quad x(x'-iz')+y-y' = 0^*,$$

by which spheres in space x', y', z' are transformed to straight lines in space x, y, z, and point spheres to straight lines of the linear complex

$$(2) \qquad dz+ydx-xdy = 0,$$

we should expect to obtain the projective group (1) of § 300, for which the linear complex (2) is an invariant.

It may be verified that this is the case, and therefore the groups (1) of § 300 and (1) of this article have the same structure.

§ 306. We have now found all possible types of primitive groups of space; that all these eight groups are primitive is easily proved; the groups (1), (2), and (3) are primitive because they have no invariant linear element for the group of the origin, a point of general position; the group (1) has been proved primitive; and the groups (5), (6), (7), and (8) are

* These are obtained from the equations of Chapter XVII by the substitution

$$y' = z'_1, \quad z' = -y'_1, \quad x' = x'_1,$$
$$x = -x_1, \quad y = y_1, \quad z = -z_1.$$

primitive because the three operators of the first degree do not leave any linear element through the origin at rest.

Collecting the results of this chapter we conclude that every primitive group of space is of one of the following types:

(1) $[p,\ q,\ r,\ xp,\ yp,\ zp,\ xq,\ yq,\ zq,\ xr,\ yr,\ zr,$
$x^2p+xyq+xzr,\ xyp+y^2q+yzr,\ xzp+yzq+z^2r]$;

(2) $[p,\ q,\ r,\ xp,\ yp,\ zp,\ xq,\ yq,\ zq,\ xr,\ yr,\ zr]$;

(3) $[p,\ q,\ r,\ xq,\ xp-yq,\ yp,\ zp,\ zq,\ xp-zr,\ xr,\ yr]$;

(4) $[p-yr,\ q+xr,\ r,\ xq,\ xp-yq,\ yp,\ xp+yq+2zr,$
$zp-y(xp+yq+zr),\ zq+x(xp+yq+zr),\ z(xp+yq+zr)]$;

(5) $[p,\ q,\ r,\ yr-zq,\ zp-xr,\ xq-yp]$;

(6) $[q+xr,\ yq+zr,\ (xy-z)p+y^2q+yzr,\ p+yr,\ xp+zr,$
$x^2p+(xy-z)q+zxr]$;

(7) $[p,\ q,\ r,\ yr-zq,\ zp-xr,\ xq-yp,\ xp+yq+zr]$;

(8) $[p,\ q,\ r,\ xq-yp,\ yr-zq,\ zp-xr,\ U,\ 2xU-S.p,$
$2yU-S.q,\ 2zU-S.r]$,

where $U=xp+yq+zr$ and $S=x^2+y^2+z^2$.

CHAPTER XXV*

SOME LINEAR GROUPS CONNECTED WITH HIGHER COMPLEX NUMBERS

§ 307. In this chapter we shall explain briefly an interesting connexion between the theory of higher complex numbers and that of a particular class of linear homogeneous groups.

(1) Let $$x'_s = \sum^{k=i=n} a_{sik}\, x_i y_k, \qquad (s = 1, \ldots, n)$$

be the finite equations of a simply transitive linear group, characterized by the property of involving the parameters $y_1, \ldots, y_n$ linearly in the finite equations of the group.

We may suppose that the coordinates have been so chosen that $(1, 0, 0, \ldots)$ is a point of general position, and therefore, the group being transitive, we may transform this point to any arbitrarily selected point by a transformation of the group; it is therefore necessary that the n linear functions

$$\sum^{k=n} a_{sik}\, y_k, \qquad (s = 1, \ldots, n)$$

should be independent.

If we now introduce a new set of parameters $z_1, \ldots, z_n$ defined by

$$z_s = \sum^{k=n} a_{sik}\, y_k,$$

the equations of the group will take the form

$$(2) \qquad x'_s = \sum^{k=i=n} \beta_{sik}\, x_i z_k;$$

and, since the coefficient of x_1 must be z_s, we shall have

$$(3) \qquad \beta_{s1k} = \epsilon_{sk},$$

where ϵ_{sk} is equal to unity if $s = k$, and to zero otherwise.

* In this chapter I have made much use of §§ 3, 4 in Chapter XXI of Lie-Scheffers' *Vorlesungen über continuierliche Gruppen.*

The equations (2) define a group which will, we assume, contain the identical transformation. It must, therefore, be possible to find $z_1, \ldots, z_n$ to satisfy the equations

$$\sum^{k=n} \beta_{sik} z_k = \epsilon_{si};$$

and in particular, taking i to be unity, to satisfy the equations

$$\sum^{k=n} \epsilon_{sk} z_k = \epsilon_{s1},$$

so that $z_1 = 1$, $z_2 = 0, \ldots, z_n = 0$, and $\beta_{si1} = \epsilon_{si}$.

Expressing the fact that the operation, resulting from first carrying out the operation with the parameters $z_1, \ldots, z_n$, and then that with the parameters $z'_1, \ldots, z'_n$, must be the same as the operation with some parameters $z''_1, \ldots, z''_n$, we have

$$(4) \quad \sum^{i=j=k=l=n} \beta_{sik} \beta_{ijl} x_j z_i z'_k = \sum^{k=i=n} \beta_{sik} z''_k x_i, \qquad (s = 1, \ldots, n).$$

Equating the coefficient of x_1 on each side we see by (3) that

$$(5) \qquad z''_s = \sum^{i=k=l=n} \beta_{sik} \beta_{s1l} z_i z'_k = \sum^{i=k=n} \beta_{sik} z_i z'_k.$$

These equations give the parameters $z''_1, \ldots, z''_n$; and if we substitute their values on the right of the equation (4), and then equate the coefficients of the variables on each side we obtain, as the necessary and sufficient conditions (in addition to $\beta_{s1k} = \beta_{sk1} = \epsilon_{sk}$) in order that (2) may be the equations of a group

$$(6) \qquad \sum^{i=n} \beta_{sik} \beta_{ijz} = \sum^{i=n} \beta_{sji} \beta_{ilk}$$

for all values of s, k, j, l from 1 to n inclusive.

A linear group of the form (1) when thrown into the form (2) is said to be in *standard* form; from (5) we see that the group in standard form is its own parameter group.

By interchanging k and j in (2) we see that the equations

$$(7) \qquad x'_s = \sum^{k=i=n} \beta_{ski} x_i z_k, \qquad (s = 1, \ldots, n)$$

also define a linear group in standard form, and with the parameters only involved linearly.

The condition that the linear transformations

$$x'_i = \sum^{j=n} a_{ij} x_j \quad \text{and} \quad x'_i = \sum^{j=n} b_{ij} x_i$$

may be permutable is

$$(8) \qquad \sum^{j=n} a_{ij}\, b_{jk} = \sum^{j=n} b_{ij}\, a_{jk};$$

we therefore see from (6) that every operation of (2) is permutable with every operation of (7); the two groups are then reciprocal.

§ 308. Conversely, any simply transitive linear group, whose reciprocal group is also linear, must be of the form (2) of § 307. We prove this as follows:

If $S_1, \ldots, S_r$ are a number of linear transformations (which need not form a group), we say that the linear transformation

$$\lambda_1 S_1 + \ldots + \lambda_r S_r,$$

where $\lambda_1, \ldots, \lambda_r$ are constants, is *dependent* on $S_1, \ldots, S_r$.

It is clear that in n variables there cannot be more than n^2 independent linear transformations.

If we are given r linear transformations $S_1, \ldots, S_r$ we cannot in general find a linear transformation T permutable with each of them; the forms of the given transformations, however, may be such that there are a number of linear transformations permutable with them.

Let $T_1, \ldots, T_s$ be the totality of all independent linear transformations permutable with $S_1, \ldots, S_r$. The condition that two linear transformations should be permutable shows us that every linear transformation dependent on $T_1, \ldots, T_s$ is permutable with every linear transformation dependent on $S_1, \ldots, S_r$. Now $T_i T_j$ is linear and permutable with $S_1, \ldots, S_r$; it must therefore be dependent upon $T_1, \ldots, T_s$, and therefore, from first principles, $T_1, \ldots, T_s$ form a finite continuous group into which the parameters enter linearly.

The operations $S_1, \ldots, S_r$ must now be operations of a linear group of the class we are now considering. For $S_i S_j$ is a linear transformation, permutable with $T_1, \ldots, T_s$; and therefore from $S_1, \ldots, S_r$ we can generate a group which will be linear, permutable with $T_1, \ldots, T_s$, and will include amongst its operations $S_1, \ldots, S_r$.

The two groups $S_1, S_2, \ldots$ and $T_1, T_2, \ldots$ will be permutable and each will involve the parameters linearly.

Let $S_1, \ldots, S_n$ be a simply transitive linear group G, with the special property that its reciprocal group Γ (which is of course simply transitive) is also linear in the variables. By what we have proved Γ must involve the parameters linearly;

and therefore G being the reciprocal group of Γ must do likewise; and therefore be of the form (7) of § 307.

§ 309. The linear operators of (2) § 307 are given by

$$X_k = \sum^{s=i=n} \beta_{sik} x_i \frac{\partial}{\partial x_s}, \qquad (k = 1, \ldots, n),$$

and in particular the group contains

$$X_1 = \sum^{s=n} x_s \frac{\partial}{\partial x_s},$$

which is permutable with every other linear operator.

A linear group therefore in which the parameters enter linearly must always contain the Abelian operator

$$\sum^{s=n} x_s \frac{\partial}{\partial x_s}.$$

If we are given the infinitesimal operators of a simply transitive linear group we may at once determine whether or not it belongs to the class of groups we are here considering. Let these operators be

$$X_k = \sum^{s=i=n} a_{sik} x_i \frac{\partial}{\partial x_s}, \qquad (k = 1, \ldots, n);$$

then, if the group is of the required class, we know that the finite transformations must be given by

$$x'_s = \sum^{i=k=n} a_{sik} x_i y_k,$$

and therefore if, and only if, these equations generate a group, will the given group be of the required class.

§ 310. We shall now determine all possible groups of this class in three variables.

First we shall prove that the alternant of two linear operators can never be equal to the linear operator

$$U \equiv \sum^{s=n} x_s \frac{\partial}{\partial x_s}.$$

The operators of the general linear homogeneous group are $x_i \dfrac{\partial}{\partial x_k}, \ldots,$ where i and k are any integers from 1 to n;

and the operators of the special linear group are $x_i \frac{\partial}{\partial x_k}, \ldots,$ where i and k are unequal, and also $x_i \frac{\partial}{\partial x_i} - x_k \frac{\partial}{\partial x_k}$.

This operator U cannot then belong to the special linear group; the alternant therefore of two operators of the special linear group can never be equal to U.

Now if X is any linear operator whatever, we can find a constant λ making $X + \lambda U$ an operator of the special linear group. We then take (X and Y being any two linear operators) $X + \lambda U$ and $Y + \mu U$ to be two operators of this special group. We have to prove that (X, Y) cannot be equal to U; if it were equal to U then $(X + \lambda U, Y + \mu U)$, being identically equal to (X, Y), would be equal to U; and we have just proved that this is impossible.

Let now X, Y, U be the operators of a group of the required class, viz. one in which the parameters enter the finite equations linearly. The operator U being permutable with every linear operator, we have

$$(U, X) = 0, \quad (U, Y) = 0, \quad (X, Y) = aX + bY + cU,$$

where a, b, c are some constants. We have just proved that a and b cannot both be zero unless c is zero; if a, b, c are all zero the group has the structure

$$(1) \qquad (U, X) = 0, \quad (U, Y) = 0, \quad (X, Y) = 0.$$

Now this group is Abelian, and therefore, if linear, must be of the required class; for its reciprocal group coincides with it, and is therefore linear, and by § 308 must therefore involve the parameters linearly in its finite equations.

If a and b are not both zero, and we take operators of the form $X + \lambda U$, $Y + \mu U$, and U as fundamental operators of the group, we can cause c to disappear from the structure constants; and we then see that fundamental operators may be so chosen that the group will have the structure

$$(2) \qquad (U, X) = 0, \quad (U, Y) = 0, \quad (X, Y) = X.$$

From what we have proved in § 263, we see that any linear operator in the variables x, y, z must be of one of the following types:

$$xp + byq + cU, \text{ where } b \text{ and } c \text{ are constants and } b \neq 1;$$

$$(3) \qquad xp + ezq + cU, \text{ where } e \text{ is zero or unity;}$$

$$e_1 xp + e_2 zq + cU, \text{ where } e_1 \text{ and } e_2 \text{ are unity or zero.}$$

We therefore can take X to be of one of the following types (since the group has U as one of its operators):

(4) $xp+byq$, where b is neither zero nor unity;

(5) xp; (6) $xp+zq$; (7) $yp+zq$; (8) zq.

We must then find Y from the identity $(X, Y)=0$, or from $(X, Y)=X$.

Let the third operator of the group be

$$Y=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta\frac{\partial}{\partial z},$$

where ξ, η, ζ are linear and homogeneous functions which can be found from the structure constants when we know X; in finding Y we may omit any part which is dependent on X and U.

Take X in the form (4) and form its alternant with Y; we have

$$x\xi_1+by\xi_2-\xi=\lambda\xi,\quad x\eta_1+by\eta_2-b\eta=\lambda by,\quad x\zeta_1+by\zeta_2=0,$$

where λ is zero if the group is in Class (1) and unity if in Class (2); we then find that the only possible group is in the first class and is

(A) $xp,\quad yq,\quad zr.$

Taking X in the form (5), we see that the group must contain $yq+zr$; and, if it is in Class (1), Y must be of the form

$$(a_1y+a_2z)q+(a_3y+a_4z)r.$$

Omitting the part $yq+zr$ we can reduce this, by § 263, to one of the two forms $yq-zr$ or zq; the group is therefore either of the form

(B) $xp,\quad zq,\quad xp+yq+zr,$

or it is of the form (A).

It may be shown that there is no group in Class (2) with X in the form (5).

It may also be verified that (6) does not lead to a new group.

Passing to (7), we see that in Class (1) Y must be of the form zp; if Y is in Class (2) it may be reduced to the form $xp-zr$ by a linear transformation.

We therefore have the two groups

(C) $\qquad yp+zq, \quad zp, \quad xp+yq+zr;$

(D) $\qquad yp+zq, \quad xp-zr, \quad xp+yq+zr.$

We next take X to be zq; if the group is in Class (1), we have

$$Y=(a_1x+a_2z)\,p+a_3xq.$$

We cannot have $a_1=a_2=0$, for this would make the group intransitive.

If $a_1=0$ but neither a_2 nor a_3 is equal to zero, we have the type (C) again.

If $a_1=a_3=0$ we get the type

(E) $\qquad zp, \quad zq, \quad xp+yq+zr.$

If a_1 is not zero, we may reduce (by linear transformation) Y to the form a_1xp; we thus obtain the type (B) again.

If the group is in Class (2) and $X=zq$, we have

$$Y=(a_1x+a_2z)\,p+(y+a_3x)\,q.$$

If $a_1=0$, then, the group being transitive, a_2 cannot be zero; by a transformation of the form

$$x'=x+\nu z, \quad y'=y+\lambda x, \quad z'=z,$$

we may then reduce Y to the form $yq+zp$.

This gives the group

(F) $\qquad zq, \quad yq+zp, \quad xp+yq+zr.$

If $a_1=1$, we may so transform that

$$Y=a_3xq-zr;$$

if a_3 is not zero, this gives the group

(G) $\qquad zq, \quad xq+zr, \quad xp+yq+zr;$

if a_3 is zero, we have the group

(H) $\qquad zq, \quad xp+yq, \quad zr.$

If a_1 is neither zero nor unity, we may reduce Y to the form

$$axp+yq;$$

and we then have the group

(I) $\qquad zq, \quad axp+yq, \quad xp+yq+zr,$

where a is neither zero nor unity.

§ 311. We must now examine all these groups to see whether the parameters occur linearly in the finite equations of the groups.

The finite equations corresponding to (A) are

$$x' = e_1 x, \quad y' = e_2 y, \quad z' = e_3 z.$$

The point (1, 0, 0) is not, however, a point of general position, since the coefficients of x in the three equations are not independent linear functions of the parameters.

These equations clearly form a group with the property of being its own parameter group. The group is not, however, in what we have defined as standard form, though it can be brought to that form. To bring it to standard form it is necessary to transform it so that in the new coordinates the point (1, 0, 0) may be one of general position. We therefore take

$$x_1 = x, \quad x_2 = x + y, \quad x_3 = x + z,$$

$$y_1 = e_1, \quad y_2 = e_1 - e_2, \quad y_3 = e_1 - e_3,$$

and thus obtain the group

$$\text{(A)} \quad x_1' = y_1 x_1, \quad x_2' = y_2 x_1 + (y_1 - y_2) x_2, \quad x_3' = y_3 x_1 + (y_1 - y_3) x_3.$$

This group is one of the class required and is in standard form.

The finite equations which correspond to (B) are

$$x' = (e_1 + e_3) x, \quad y' = e_3 y + e_2 z, \quad z' = e_3 z.$$

If we take

$$x_1 = z, \quad x_2 = y, \quad x_3 = x + z,$$

$$y_1 = -e_3, \quad y_2 = e_2, \quad y_3 = e_1,$$

we have a group of the required class

$$\text{(B)} \quad x_1' = y_1 x_1, \quad x_2' = y_2 x_1 + y_1 x_2, \quad x_3' = y_3 x_1 + (y_1 - y_3) x_3.$$

The operators (C) lead to the group

$$\text{(C)} \quad x_1' = y_1 x_1, \quad x_2' = y_2 x_1 + y_1 x_2, \quad x_3' = y_3 x_1 + y_2 x_2 + y_1 x_3,$$

which is of the required class and in standard form.

If the operators (D) lead to a group whose finite equations involve the parameters linearly, the equations in finite form must be

$$x' = (e_2 + e_3) x + e_1 y, \quad y' = e_3 y + e_1 z, \quad z' = (e_3 - e_2) z.$$

Now these are not the equations of a group at all, so that the equations (D) do not lead to a group of the type we want.

Similarly we see that (F), (G), and (I) do not lead to the required type of group.

The operators (E) lead to

(E) $x_1' = y_1 x_1, \quad x_2' = y_2 x_1 + y_1 x_2, \quad x_3' = y_3 x_1 + y_1 x_3.$

Finally the operators (H) lead to

(H) $x_1' = y_1 x_1, \; x_2' = y_2 x_1 + y_1 x_2 + y_2 x_3, \; x_3' = y_3 x_1 + (y_1 + y_3) x_3.$

There are, therefore, only five types of groups in three variables which are linear in both variables and parameters; and of these groups only (H) is non-Abelian.

An example of a non-Abelian group linear in four variables and four parameters is

$$
\begin{aligned}
x_1' &= y_1 x_1 - y_2 x_2 - y_3 x_3 - y_4 x_4,\\
x_2' &= y_2 x_1 + y_1 x_2 - y_4 x_3 - y_3 x_4,\\
x_3' &= y_3 x_1 - y_4 x_2 + y_1 x_3 + y_2 x_4,\\
x_4' &= y_4 x_1 + y_3 x_2 - y_2 x_3 + y_1 x_4.
\end{aligned}
$$

An example of an Abelian linear group in five variables is

$$
\begin{aligned}
x_1' &= y_1 x_1,\\
x_2' &= y_2 x_1 + y_1 x_2,\\
x_3' &= y_3 x_1 + y_2 x_2 + y_1 x_3,\\
x_4' &= y_4 x_1 + y_3 x_2 + y_2 x_3 + y_1 x_4,\\
x_5' &= y_5 x_1 + y_4 x_2 + y_3 x_3 + y_2 x_4 + y_1 x_5 *.
\end{aligned}
$$

§ 312. We now proceed to explain the connexion of these results with the theory of higher complex numbers.

Let $e_1, \ldots, e_n$ be a system of n independent complex numbers; any number x of the system can be expressed in the form

$$x = x_1 e_1 + \ldots + x_n e_n,$$

where $x_1, \ldots, x_n$ are ordinary numbers; x can therefore only be equal to zero when $x_1, \ldots, x_n$ are each zero.

We call $e_1, \ldots, e_n$ the fundamental complex numbers of the system; but if $\beta_1, \ldots, \beta_n$ are any n independent complex numbers of the system we could equally take them to be the fundamental complex numbers, and express all other numbers in terms of them.

From the fact that the number resulting from the multiplication of two complex numbers must be expressible in terms of the fundamental complex numbers we have

* Burnside, *Proceedings of the London Mathematical Society*, XXIX, p. 339.

$$e_k e_i = \sum^{j=n} \gamma_{jki} e_j,$$

where $\gamma_{jki}, \ldots$ are a system of ordinary numbers, fixed when we have chosen our fundamental complex numbers. If, therefore, u is the complex number yx,

$$u_s = \sum^{i=k=n} \gamma_{ski} y_k x_i.$$

Similarly, if v is the complex number xy,

$$v_s = \sum^{i=k=n} \gamma_{sik} y_k x_i.$$

From the fact that division is to be an operation possible in the system—that is, when we are given x and u, or x and v, we must be able in general to determine y—we see that the determinant M_x whose s[th] row and k[th] column is

$$\sum^{i=n} \gamma_{ski} x_i,$$

cannot vanish identically; nor can the determinant M'_x, whose s[th] row and k[th] column is $\sum\limits^{i=n} \gamma_{sik} x_i$, vanish identically.

It follows, therefore, that the equation system

$$(1) \qquad x'_s = \sum^{i=k=n} \gamma_{ski} y_k x_i, \qquad (s = 1, \ldots, n),$$

where we look on $x_1, \ldots, x_n$ as the original variables, and $x'_1, \ldots, x'_n$ as the transformed, is such that the determinant of the transformation does not vanish.

For a similar reason the determinant of

$$(2) \qquad x'_s = \sum^{i=k=n} \gamma_{sik} y_k x_i$$

does not vanish.

Since in the system of complex numbers the law of multiplication is to be associative, if $u = yx$ and $v = zy$, we must have $zu = vx$. Therefore

$$\sum^{t=i=n} z_i u_t \gamma_{sit} e_s = \sum^{t=k=n} v_t x_k \gamma_{stk} e_s; \text{ and therefore}$$

$$\sum^{t=i=j=k=n} z_i e_s \gamma_{sit} \gamma_{tjk} y_j x_k = \sum^{t=i=j=k=n} x_k \gamma_{stk} e_s \gamma_{tij} z_i y_j.$$

Equating the coefficients of $z_i e_s x_k y_j$ on each side we have

$$(3) \qquad \sum^{t=n} \gamma_{sit}\gamma_{tjk} = \sum^{t=n} \gamma_{stk}\gamma_{tij}.$$

Now these are just the conditions that (1) should generate a group which is its own parameter group, and they are equally the conditions that (2) should do so.

§ 313. We must now prove that these groups contain the identical transformation.

Let $x = x_1e_1 + \ldots + x_ne_n$ be a general complex number, that is, a number such that neither M_x nor M'_x is zero; we can, whatever u may be, find a complex number y such that u is equal to yx. Now let u be taken equal to x, and let the corresponding number y be denoted by ϵ, so that x is equal to ϵx; we shall prove that ϵ does not depend on x at all, and shall investigate its position in the system.

Let v be any other general complex number, and z a complex such that v is equal to xz; we have

$$\epsilon v = \epsilon xz = xz = v;$$

that is, ϵ has the same relation to v as to x, and therefore does not depend on either v or x.

Next we see that if yx is zero, where x is a general complex number, we must have, since M_x is not zero,

$$y_1 = 0, \ldots, y_n = 0.$$

So, since M'_x is not zero, if xy is zero, we must have

$$y_1 = 0, \ldots, y_n = 0.$$

Let x' be equal to $x\epsilon$, then

$$x'x = x\epsilon x = xx,$$

and therefore $(x' - x)x$ is zero, so that x' is equal to x; that is, we also have $x = x\epsilon$.

This unique number ϵ is therefore a complex unity.

Let $\epsilon = \epsilon_1 e_1 + \ldots + \epsilon_n e_n$, where $\epsilon_1, \ldots, \epsilon_n$ are ordinary numbers, then, since $x = x\epsilon = \epsilon x$, we have

$$x_s = \sum^{i=k=n} \gamma_{sik} x_i \epsilon_k = \sum^{i=k=n} \gamma_{ski} \epsilon_k x_i.$$

We now see that $\quad y_k = \epsilon_k, \qquad (k = 1, \ldots, n)$

will give the identical transformation in (1) and (2) of § 312.

The two equation systems, therefore, define groups each containing the identical transformation; and, since neither

M_x nor M'_x is zero, there are n effective parameters; that is, the groups are simply transitive, and involve the parameters linearly, and each group has the property of being its own parameter group.

If we were to take ϵ as one of our fundamental complex numbers, say e_1, we should have each group in its standard form.

§ 314. The infinitesimal operators of (1), §312, are $X_1, \ldots, X_n$, where

$$X_k = \sum^{i=s=n} \gamma_{sik} x_i \frac{\partial}{\partial x_s}$$

and

$$(X_i, X_k) = \sum^{j=n} c_{ikj} X_j.$$

Now

$$(X_i, X_k) = \sum^{s=t=j=n} (\gamma_{sji}\gamma_{tsk} - \gamma_{sjk}\gamma_{tsi}) x_j \frac{\partial}{\partial x_t},$$

$$= \sum^{s=t=j=n} (\gamma_{sik} - \gamma_{ski}) \gamma_{tjs} x_j \frac{\partial}{\partial x_t}, \text{ by (3) of § 312,}$$

$$= \sum^{s=n} (\gamma_{sik} - \gamma_{ski}) X_s;$$

and therefore $c_{iks} = \gamma_{sik} - \gamma_{ski}$.

Similarly we may write down the operators of the group (2) of § 312; and it may be at once verified (by aid of (3) § 312) that the two sets of operators are permutable, so that the groups are reciprocal.

We thus see that to every system of complex numbers there will correspond two simply transitive reciprocal linear groups; and conversely, to every pair of such groups a system of complex numbers.

The complex number ϵ whose existence we have proved may be taken to be an ordinary unit number since $\epsilon x = x\epsilon = x$. The fundamental complex numbers may therefore be taken to be the ordinary unity and $e_2, \ldots, e_n$ as in the Hamiltonian Quaternion system.

§ 315. When we are given a simply transitive linear group in standard form, and wish to write down the corresponding system of complex numbers, we multiply x'_1 by e_1, x'_2 by e_2, ... and, adding, equate the coefficient of $x_i y_k$, on the right of the transformation scheme, to $e_i e_k$.

The laws of combination of the symbols $e_1, \ldots, e_n$ are most conveniently expressed in the form of a square of n^2 com-

partments, the expression equal to $e_i e_k$ being found in the compartment corresponding to the i^{th} row and k^{th} column.

Thus the system corresponding to (H) is denoted by

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	0	0
e_3	e_3	0	e_3

;

this means that

$$e_1^2 = e_1, \quad e_2^2 = 0, \quad e_3^2 = e_3, \quad e_1 e_2 = e_2, \quad e_2 e_1 = e_2,$$
$$e_1 e_3 = e_3, \quad e_3 e_1 = e_3, \quad e_2 e_3 = 0, \quad e_3 e_2 = e_2,$$

where we understand that the operation on the right in $e_i e_k$ is to be taken first.

The other systems in three complex numbers are all commutative, since the groups are Abelian.

The non-Abelian group of order four gives the system

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	$-e_1$	e_2
e_4	e_4	e_3	$-e_2$	$-e_1$

;

i.e. the Hamiltonian Quaternion system, when we take $e_1 = 1$.

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